Age of Information of Jackson Networks with Finite Buffer Size

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***Abstract*—We investigate the Age of Information (AoI) of Jackson Networks with finite buffer size. We provide a closed form expression of an upper bound of the average AoI by using Stochastic Hybrid Systems (SHS) along with linear algebra tools. Since the computation of the bound requires inverting a $N \times N$ matrix, where N is the number of queues in the network, we provide a simple iterative algorithm and prove its convergence with geometric convergence rate. Numerical results allow us to assess the accuracy of our model in various settings.**

***Index Terms*—Age of Information, Jackson networks.**

I. Introduction

The notion of freshness of information is gaining interest in many areas, e.g. control, communications networks, etc. This is mainly motivated by the development of services in which a monitor tracks fresh updates about a physical process. In order to capture the timeliness of information, a relatively new metric called Age of Information (AoI) has been introduced in [1], [2].

AoI has attracted the attention of researchers in the past few years. Most of the work has focused on the computation of the average AoI and its minimization in different settings, where the medium (channel, network, etc.) between the transmitter and the monitor is modeled as a queueing system. For instance, M/M/1, M/D/1 and D/M/1 queue models have been considered in [2]. Furthermore, the AoI has been analyzed in [3], [4] by considering an M/M/2 queue model. Other queueing models have also been studied in [5], [6], [7]. The aforementioned works focused on single hop networks with predefined transmission/scheduling policy. The problem of scheduling and access in single hop networks has also been explored, and optimal and near-optimal policies have been obtained in various scenarios [8], [9], [10], [11]. Unfortunately, single hop and/or single server queueing models may not be representative of the networks where packets can be sent through multiple paths. Therefore, AoI has been also analyzed in networks with parallel servers [12], [13], [14] as well as in multi-hop systems [15], [16]. In [15], [16], the scheduling in multi-hop queueing networks has been considered and it was proved that Preemptive Last Generated First Served (P-LGFS) policy is age-optimal for i.i.d. exponentially distributed service times. The authors in [17], [18] characterized the average AoI in a system of tandem queues without buffer. In fact, [17] considered two non-preemptive queues, whereas in [18], the author considered the preemptive case in a system with n tandem queues without buffer. The authors in [19] considered a specific multi-hop scenario in which each node is both a source and a monitor, derived fundamental age limits and developed near optimal scheduling policies. Note that the multi-hop multi-cast setting has also been considered in [20]. The works in [17], [18] applied the Stochastic Hybrid Systems (SHS) method, which was first used to analyze the average AoI in in [21]. The SHS approach models the system as a combination of continuous and discrete states; the former state tracks the age evolution in the system while the latter one is a discrete process that captures the evolution of the system in question and is usually represented by a Markov Chain. For more comprehensive review of recent work on AoI, one can refer to [22]. In this paper, we study the AoI in Jackson networks. A Jackson network models a network of queues in which the packets are routed from one queue to another which allows us to model the performance/behavior of packets in complex networks. In Jackson networks, it is shown that the steady-state distribution of packets in the queues admits a product form solution [23], which is possibly one of the most important results in queueing theory. Much research has since been devoted to investigating queueing networks in which product form solutions exist (see for instance [24]). We provide a closed form expression of an upper bound on the average AoI of Jackson networks with FIFO queues by using the SHS technique. Since the closed form expression requires inverting a matrix of size $N \times N$ (and has hence a computational complexity of $\mathcal{O}(N^{3.5})$), we provide a simple iterative algorithm and prove that it converges with a geometric convergence rate. We also provide numerical results that assess the accuracy of the derived bound.

II. Model Description

We study the average AoI in a system formed by N queues. Jobs arrive to queue i according to a Poisson process of parameter λ_i and jobs in queue i are served with exponential time of rate μ_i. We assume that the queues have a finite buffer size. We denote by $M_i \geq 0$ the buffer size of queue i. When a job arrives to queue i and there are M_i jobs waiting for service in that queue, the arriving job replaces the job in the last position of the queue. When a job is served in queue i, it leaves the system with probability $p_{i,s}$ and it is routed to

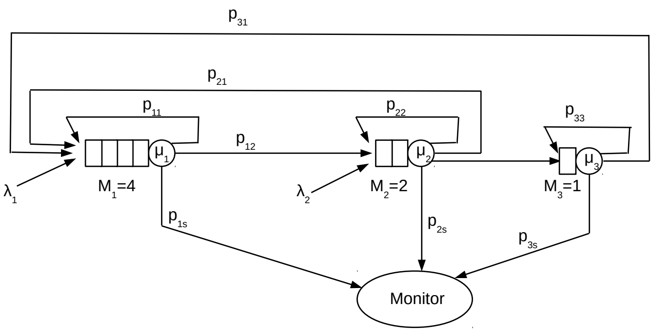

Fig. 1: Example of a Jackson network with three nodes.

queue j with probability $p_{i,j}$. Therefore,

$$p_{i,s} + \sum_{j=1}^{N} p_{i,j} = 1,$$

for all i. We consider that the jobs are served in the queues according to the FIFO discipline. Although LIFO may provide better age results, FIFO is more used in practice, especially when the servers are usually used to deliver multiple services and not only status updates. Analyzing the average AoI in the network in question using the SHS method is very complicated, since it requires tracking the state of each node, which is the same as handling a Markov Chain where the number of states is equal to $\prod_{i=1}^{N}(M_i+1)$. We therefore provide a modified system in the next section, allowing us to derive a closed form expression of a bound of the average AoI.

III. Modified System

We consider a system where we use fake updates to simplify the computation of the average AoI of the aforementioned system. More precisely, we consider that the queues are full of updates and, when a job is served, all the packets in the queue are moved forward one position and, in the last position, we put a fake update whose value is the same as the age of the penultimate packet. By doing so, the Markov Chain to be considered in the SHS method has a single state. [1]

In this section, we characterize the average AoI for the modified system with N nodes by using the SHS method. A closed form expression of the average AoI is provided. We then provide a simple iterative method in order to obtain the average AoI with reduced computational complexity.

Remark 1: It is important to note that the average AoI of the modified system provides an upper bound for that of the original system. In fact, in the modified system, the buffer is always full and contains fake packets while in the original system the buffer may not be full all the time. In the modified system, the incoming update/packet is always put in the last position of the queue, which increases the sojourn time of the packets in the queue and hence leads to a larger average AoI as compared to the original system.

[1]It is implicitly assumed that, if there exists a queue which is not initially full of packets, we put fake updates until the buffer is full.

A. Average AoI of the Modified System

We provide in this subsection a derivation of the average AoI for the modified system by making use of SHS and linear algebra tools.

Proposition 1: In the aforementioned system, the average AoI is

$$\Delta = \frac{1}{\sum_{j=1}^{N}\mu_j p_{j,s}}\left(1+\sum_{j=1}^{N} M_j p_{j,s} + \sum_{j=1}^{N} p_{j,s}\mu_j y_j\right), \quad (1)$$

where $y_1,\dots,y_N$ satisfy that

$$y_i\left(\lambda_i + \sum_{\substack{j=0\\ j\neq i}}^{N}\mu_j p_{j,i}\right) = 1 + \sum_{j=1}^{N} M_j p_{j,i} + \sum_{\substack{j=0\\ j\neq i}}^{N} y_j\mu_j p_{j,i}, \quad (2)$$

with $i = 1,\dots,n$.

Proof: The proof is provided in Appendix A. ■

In the following, we provide a closed form expression of the average AoI of the considered network. For that, we introduce the following notations. Let $\mathbf{F}$ be a $N\times N$ matrix whose element in the i-th row and j-th column is defined as follows:

$$F_{i,j} = \begin{cases} \frac{\mu_j p_{j,i}}{\lambda_i + \sum_{\substack{j=0\\ j\neq i}}^{N}\mu_j p_{j,i}} & \text{if } i\neq j,\\ 0 & \text{if } i = j. \end{cases} \quad (3)$$

Furthermore, we define

$$b_i = \frac{1+\sum_{j=1}^{N} M_j p_{j,i}}{\lambda_i + \sum_{\substack{j=0\\ j\neq i}}^{N}\mu_j p_{j,i}}$$

for $i\in\{1,...,N\}$. The system of equations in (2) can be written in the following matrix form

$$(\mathbf{I}-\mathbf{F})\,\mathbf{y} = \mathbf{b}$$

where $\mathbf{y} = [y_1,...,y_N]^T$ and $\mathbf{b} = [b_1,...,b_N]^T$.

Proposition 2: The unique solution to the system of equations in (2) is given by

$$\mathbf{y}^* = (\mathbf{I}-\mathbf{F})^{-1}\mathbf{b} \quad (4)$$

Proof: The proof is provided in Appendix B. ■

Corollary 1: The average AoI of the modified system is given by

$$\Delta = \frac{1}{\sum_{j=1}^{N}\mu_j p_{j,s}}\left(1+\sum_{j=1}^{N} M_j p_{j,s} + \mathbf{p}_\mu{}^T(\mathbf{I}-\mathbf{F})^{-1}\mathbf{b}\right), \quad (5)$$

where $\mathbf{p}_\mu = [(p_\mu)_1,\dots,(p_\mu)_N]^T$, where $(p_\mu)_i = p_{i,s}\cdot\mu_i$ for all $i=1,\dots,N$.

Proof: The expression can obtained by replacing the expression $\mathbf{y}^* = (\mathbf{I}-\mathbf{F})^{-1}\mathbf{b}$ in (1). ■

B. Iterative Algorithm

The closed form expression of the average AoI requires the computation of the inverse of matrix $\mathbf{I}-\mathbf{F}$ (of dimension $N \times N$), which has a complexity of order $\mathcal{O}(N^{3.5})$. In order to compute the average AoI Δ with a reduced complexity, we can use the following iterative algorithm

$$\mathbf{y}(t+1) = \mathbf{F}\mathbf{y}(t) + \mathbf{b} \tag{6}$$

where $\mathbf{y}(t)$ is $\mathbf{y}$ obtained at iteration t. This algorithm converges to $\mathbf{y}^* = (\mathbf{I}-\mathbf{F})^{-1}\mathbf{b}$ with a geometric convergence rate. This will be discussed and proved later on in this section. Once the algorithm in (6) converges to $\mathbf{y}^*$, the average AoI can be computed using (1), i.e.

$$\Delta = \frac{1}{\sum_{j=1}^{N} \mu_j p_{j,s}} \left(1 + \sum_{j=1}^{N} M_j p_{j,s} + \mathbf{p}_\mu{}^T \mathbf{y}^*\right), \tag{7}$$

Before showing the convergence of the aforementioned iterative algorithm in (6), we provide the following definition.

Definition 1: The sequence $\{\mathbf{y}(1), \mathbf{y}(2), ...\}$ is said to converge geometrically to $\mathbf{y}^*$ if $\|\mathbf{y}(t) - \mathbf{y}^*\|_\infty \leq C\alpha^t$, where $C > 0$ and $0 < \alpha < 1$ are constant values and $\|.\|_\infty$ is the infinity norm.

Proposition 3: The sequence $\{\mathbf{y}(1), \mathbf{y}(2), ...\}$ generated by the algorithm in (6) converges geometrically to $\mathbf{y}^* = (\mathbf{I}-\mathbf{F})^{-1}\mathbf{b}$

Proof: The proof is provided in Appendix C. ∎

This result implies that the iterative algorithm in (6) is efficient and converges fast. In the simulations in section IV, we have observed that it converges in a very small number of iterations (5 to 10 iterations) in the considered cases.

IV. Numerical experiments

We provide simulation results to study the accuracy of the upper bound of the average AoI with respect to the real average AoI, that is, the average AoI of the original system. We consider a network with two queues. We set $\mu_1 = \mu_2 = 1$ as well as the following routing matrix: $p_{1,1} = 0.2$, $p_{1,2} = 0.7$, $p_{1,s} = 0.1$, $p_{2,1} = 0$ and $p_{2,2} = 0.5$ and $p_{2,s} = 0.5$.

In Figure 2, we present the percentage relative error of the upper bound w.r.t. the real age for $M/M/1/1$ queues. We observe that the difference between the upper bound and the real average AoI is very small (does not exceed 4%) and close to zero for most of the considered values of λ_1 and λ_2. For example, for two M/M/1/1 queues, when $\lambda_1 = \lambda_2 = 2$, the real average AoI is 2.47 and the upper bound we provide is 2.5, which results in a relative error of 1.1%. As it can be seen in Figure 3, this phenomenon holds true as well when we consider M/M/1/2* queues. We notice that the approximation error is higher as compared to the previous case, however, the gap between the bound and the real AoI is less than 5% for most of the considered arrival rates λ_1 and λ_2 and can achieve 13% for very small values of arrival rates. This gap/error occurs because, in the modified system, the incoming updates find the system full (sometimes with fake packets in the buffer) and are always put in the last position of the buffer, which increases the sojourn time in the buffer. However, since

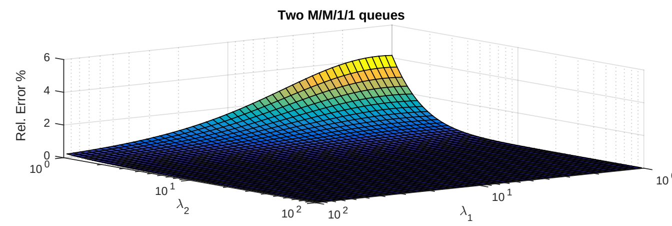

Fig. 2: Tightness (in %) of the upper bound for two M/M/1/1 queues for different values of arrival rates. X-axis and Y-axis in log scale.

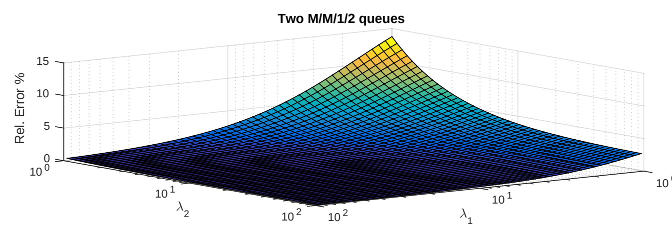

Fig. 3: Tightness (in %) of the upper bound for two M/M/1/2 queues for different values of arrival rates. X-axis and Y-axis in log scale.

there is no fake packets in the original system, there will be in general less packets in the buffer than in the modified system, which leads to a better AoI.

V. Conclusion

In this paper, we provide an analysis of AoI in a Jackson network composed of N queues with finite buffer size. We considered Poisson arrival and exponential service time distributions and that a served packet is either routed to a new queue or delivered to the monitor. By using SHS and linear algebra tools, we provided a closed form expression of an upper bound on the average AoI. Since computing this bound requires inverting a $N \times N$ matrix, we provided an iterative algorithm and proved its convergence with geometric convergence rate. We also provided numerical results to asses the accuracy of the proposed model in various settings.

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l	λ^l	x'_0	$x'_{i,0}$	$x'_{i,1}$	$\dots$	x'_{i,M_i-1}	x'_{i,M_i}	x'_{j,M_j}
0	λ_i	x_0	$x_{i,0}$	$x_{i,1}$	$\dots$	x_{i,M_i-1}	0	x_{j,M_j}
1	$\mu_j p_{j,i}$	x_0	$x_{i,0}$	$x_{i,1}$	$\dots$	x_{i,M_i-1}	$x_{j,0}$	x_{j,M_j}
2	$\mu_i p_{i,s}$	$x_{i,0}$	$x_{i,1}$	$x_{i,2}$	$\dots$	x_{i,M_i}	x_{i,M_i}	x_{j,M_j}
3	$\mu_i p_{i,j}$	x_0	$x_{i,1}$	$x_{i,2}$	$\dots$	x_{i,M_i}	x_{i,M_i}	$x_{i,0}$
4	$\mu_i p_{i,i}$	x_0	$x_{i,1}$	$x_{i,2}$	$\dots$	x_{i,M_i}	$x_{i,0}$	x_{j,M_j}

TABLE I: Table of SHS transitions of Figure 4.

Appendix A
Proof of Proposition 1

We use the SHS method to compute the average AoI of the modified system. For this case, the discrete state is formed by a single state since we have used fake updates and the continuous state is defined as

$$\mathbf{x}(t) = [x_0(t)\ \vec{x}_1(t)\ \vec{x}_2(t)\ \dots \vec{x}_N(t)],$$

where $\vec{x}_i(t) = [x_{i,0}(t)\ x_{i,1}(t)\ \dots x_{i,M_i}(t)]$ for all $i = 1, \dots, N$. This notation means that the age of the update in service in server i is $x_{i,0}$, the age of the update in the position j of queue i is $x_{i,j}$ (with $j = 1, \dots, M_i$) and the age of the monitor is x_0.

Let us remark that, for an arrival or an update delivery after being served in queue j, the age of the updates in the rest of the queues is not modified. Besides, when an update is routed to queue j, the update in the last position of the queue is the only one whose age changes. This allows us to focus on the age of the updates in queue i and of the update in the last position of the rest of the queues to illustrate the Markov Chain and the SHS transitions, which are presented respectively in Figure 4 and in Table I. We now explain each transition l:

$l = 0$ An update arrives from outside to queue i. For this case, the value of x_{i,M_i} is replaced by 0 and the other elements of $\mathbf{x}$ do not change.

$l = 1$ An update arrives to queue i from queue j. For this case, the value of x_{i,M_i} is replaced by $x_{j,0}$ and the other elements of $\mathbf{x}$ do not change.

$l = 2$ An update in queue i ends the service and it is delivered to the monitor. For this case, the value of x_0 is replaced by $x_{i,0}$ and $x_{i,k}$ by $x_{i,k+1}$, with $k = 0, \dots, M_i - 1$. The other elements of $\mathbf{x}$ do not change.

$l = 3$ An update in queue i ends the service and it is routed to queue j (with $j \neq i$). For this case, the value of x_{j,M_j} is replaced by $x_{i,0}$ and $x_{i,k}$ by $x_{i,k+1}$, with $k = 0, \dots, M_i - 1$. The other elements of $\mathbf{x}$ do not change.

$l = 4$ An update in queue i ends the service and it is routed to queue i. For this case, the value of x_{i,M_i} is replaced by $x_{i,0}$ and $x_{i,k}$ by $x_{i,k+1}$, with $k = 0, \dots, M_i - 1$. The other elements of $\mathbf{x}$ do not change.

We have a trivial stationary distribution of the Markov Chain since it is formed by a single state. We define also the vector $\mathbf{v} = [v_0\ \vec{v}_1\ \vec{v}_2\ \dots\ \vec{v}_N]$, where $\vec{v}_i = [v_{i,0}\ v_{i,1}\ \dots\ v_{i,M_i}]$ for all $i = 1, \dots, N$. From the result of Theorem 4 in [21], it follows that the average AoI is equal to v_0 and $\mathbf{v}$ satisfies the following system of equations (the vector $\mathbf{b}$ in [21] is an

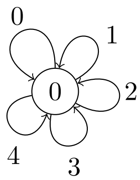

Fig. 4: The SHS Markov Chain for a Jackson Network formed by N nodes.

all-ones vector of size $N+1+\sum_{i=1}^{N} M_i$ in our case):

$$v_0 \sum_{j=1} (\lambda_j + \mu_j) = 1 + v_0 \sum_{j=1}^{N} (\lambda_j + \mu_j(1-p_{j,s})) + \sum_{i=1}^{N} v_{j,0}\mu_j p_{j,s}$$

$$v_{i,k}\mu_i = 1 + v_{i,k}\left(\sum_{j=1}^{N}\lambda_j + \sum_{j\neq i}\mu_j\right) + v_{i,k+1}\mu_i,\ \forall k, i.$$

$$v_{i,M_i}\sum_{j=1}^{N}(\lambda_j+\mu_j) = 1 + v_{i,M_i}\sum_{j\neq i}\lambda_j + v_{i,M_i}\sum_{j=1}^{N}\mu_j(1-p_{j,i}) + \sum_{j=1}^{N} v_{j,0}\mu_j p_{j,i},\ \forall i = 1,\ldots,N$$

Simplifying the above expressions, it results that

$$\begin{aligned} v_0 \sum_{j=1} \mu_j p_{j,s} &= 1 + \sum_{j=1}^{N} v_{j,0}\mu_j p_{j,s} \\ v_{i,k}\mu_i &= 1 + v_{i,k+1}\mu_i,\ \forall k = 0,\ldots,M_i-1, \forall i = 1,\ldots,N \\ v_{i,M_i}\left(\lambda_i + \sum_{j=1}^{N}\mu_j p_{j,i}\right) &= 1 + \sum_{j=1}^{N} v_{j,0}\mu_j p_{j,i},\ \forall i = 1,\ldots,N \end{aligned}$$

Since $v_{i,k}\mu_i = 1 + v_{i,k+1}\mu_j,\ \forall k = 0,1,\ldots,M_i-1$, we derive that for all i,

$$v_{i,0}\mu_i = M_i + v_{i,M_i}\mu_i.$$

Therefore, the first equation of the above system of equations can be written as,

$$v_0 \sum_{j=1} \mu_j p_{j,s} = 1 + \sum_{j=1}^{N} v_{j,0}\mu_j p_{j,s} = 1 + \sum_{j=1}^{N}(M_j + v_{j,M_j}\mu_j)p_{j,s}$$

whereas the third equation:

$$\begin{aligned} v_{i,M_i}\left(\lambda_i + \sum_{j=1}^{N}\mu_j p_{j,i}\right) &= 1 + \sum_{j=1}^{N} v_{j,0}\mu_j p_{j,i} \\ &= 1 + \sum_{j=1}^{N}(M_j + v_{j,M_j}\mu_j)p_{j,i}. \end{aligned}$$

Let $y_i = v_{i,M_i}$. Hence,

$$\begin{aligned} v_0 \sum_{j=1} \mu_j p_{j,s} &= 1 + \sum_{j=1}^{N}(M_j + y_j\mu_j)p_{j,s}, \\ y_i\left(\lambda_i + \sum_{j=1}^{N}\mu_j p_{j,i}\right) &= 1 + \sum_{j=1}^{N}(M_j + y_j\mu_j)p_{j,i}, \end{aligned}$$

for all $i = 1,\ldots,N$. Or alternatively,

$$\begin{aligned} v_0 \sum_{j=1} \mu_j p_{j,s} &= 1 + \sum_{j=1}^{N} M_j p_{j,s} + \sum_{j=1}^{N} y_j\mu_j p_{j,s} \\ y_i\left(\lambda_i + \sum_{\substack{j=0\\ j\neq i}}^{N}\mu_j p_{j,i}\right) &= 1 + \sum_{j=1}^{N} M_j p_{j,i} + \sum_{\substack{j=0\\ j\neq i}}^{N} y_j\mu_j p_{j,i}, \end{aligned}$$

for all $i = 1,\ldots,N$, and the desired result follows.

Appendix B
Proof of Proposition 2

Solving linear systems of equations has been investigated in the past in several areas. In particular, in the context of power control in wireless networks, similar system of equations have been analyzed in [25], [26]. It has been shown that, if matrix $\mathbf{F}$ is irreducible, a system of equations of the form $(\mathbf{I}-\mathbf{F})\,\mathbf{y} = \mathbf{b}$ has a unique solution $\mathbf{y}^* = (\mathbf{I}-\mathbf{F})^{-1}\,\mathbf{b}$, if and only if the spectral radius of $\mathbf{F}$ is $\rho(\mathbf{F}) < 1$. One can refer to [25], [26] for more details. Therefore, it is sufficient to show here that $\mathbf{F}$ is irreducible and $\rho(\mathbf{F}) < 1$ to prove the result. In our context here, $\mathbf{F}$ has the following special structure: all off-diagonal entries are strictly positive and $F_{i,i} = 0\ \forall\ i$. This matrix can be shown to be irreducible: the proof is simple and consists of showing that the corresponding graph to $\mathbf{F}$ is strongly connected (i.e. $\mathbf{F}$ is the adjacency matrix of the graph). In fact, one can construct a directed graph composed of N vertices, in which the weight of the edge between two vertices i and j is $F_{i,j}$. Since all $F_{i,j} \neq 0\ \forall\ i \neq j$, there is a path from any vertex i to any other vertex j. In addition, there is a path from a vertex i to itself : $i \to j$ and then $j \to i$ (since both $F_{i,j} \neq 0$ and $F_{j,i} \neq 0$). Therefore, the constructed graph is strongly connected, which implies that the corresponding adjacency matrix $\mathbf{F}$ of the graph is irreducible. The next step is to show that the spectral radius of $\mathbf{F}$ is strictly less than 1. This can be deduced from the fact that the sum of the elements in each row is < 1. In fact,

$$\sum_{j=1}^{N} F_{i,j} = \frac{\sum_{\substack{j=0\\ j\neq i}}^{N}\mu_j p_{j,i}}{\lambda_i + \sum_{\substack{j=0\\ j\neq i}}^{N}\mu_j p_{j,i}} < 1\ \forall i$$

Therefore, the following norm of $\mathbf{F}$, $\|\mathbf{F}\|_\infty = \max_i \sum_{j=1}^{N} F_{i,j} < 1$, and by using the inequality $\rho(\mathbf{F}) \leq \|\mathbf{F}\|_\infty =$ we conclude that $\rho(\mathbf{F}) < 1$. This ends the proof.

Appendix C
Proof of Proposition 3

Let $\mathbf{y}(t)$ be the obtained vector by the algorithm in (6) at iteration t.

$$\|\mathbf{y}(t) - \mathbf{y}^*\|_\infty = \|\mathbf{F}\mathbf{y}(t-1) + \mathbf{b} - \mathbf{y}^*\|_\infty$$

Recall that $\mathbf{y}^*$ is the unique solution to (1) and it is therefore a fixed point of (6), that is $\mathbf{y}^* = \mathbf{F}\mathbf{y}^* + \mathbf{b}$. We have therefore,

$$\begin{aligned}\|\mathbf{y}(t) - \mathbf{y}^*\|_\infty &= \|\mathbf{F}\left(\mathbf{y}(t-1) - \mathbf{y}^*\right)\|_\infty \\ &\leq \|\mathbf{F}\|_\infty \|\mathbf{y}(t-1) - \mathbf{y}^*\|_\infty \\ &\leq \left(\|\mathbf{F}\|_\infty\right)^t \|\mathbf{y}(0) - \mathbf{y}^*\|_\infty \end{aligned} \tag{8}$$

We know from the definition of $\mathbf{F}$ that $\|\mathbf{F}\|_\infty = \max_i \sum_{j=1}^{N} F_{i,j} < 1$ (since $\sum_{j=1}^{N} F_{i,j} < 1 \ \forall \ i$). By denoting $C = \|\mathbf{y}(0) - \mathbf{y}^*\|_\infty$ and $\alpha = \|\mathbf{F}\|_\infty$, we have,

$$\|\mathbf{y}(t) - \mathbf{y}^*\|_\infty \leq C\left(\alpha\right)^t$$

We conclude that when $t \to \infty$, $\|\mathbf{y}(t) - \mathbf{y}^*\|_\infty \to 0$ and in addition the sequence $\{\mathbf{y}(1), \mathbf{y}(2), ...\}$ generated by the algorithm in (6) converges geometrically to $\mathbf{y}^*$.