To Confine or not to Confine: A Mean Field Game Analysis of the End of an Epidemic

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Abstract. We analyze a mean field game where the players dynamics follow the SIR model. The players are the members of the population and the strategy consists in choosing the probability of being exposed to the infection, i.e., its confinement level. The goal of each player is to minimize the sum of the confinement cost, which is linear and decreasing on its strategy, and a cost of infection per unit time. We formulate this problem as a mean field game and we investigate the structure of a mean field equilibrium. We study the behavior of agents during the end of the epidemic, where the proportion of infected population is decreasing. Our main results show that: (a) when the cost of infection is low, a mean field equilibrium consists of never getting confined, i.e., the probability of being exposed to the infection is always one and (b) when the cost of infection is large, a mean field equilibrium consists of being confined at the beginning and, after a given time, being exposed to the infection with probability one.

Keywords: Mean field game · SIR model · Confinement.

1 Introduction

The situation derived from COVID19 disease has put in evidence the need of carrying out research in the epidemic field. The most important epidemic model that has been investigated in the literature is based on the SIR model. In the SIR model, it is considered that each member of the population belongs to one of the following states: susceptible (S), infected (I) or recovered (R). It has been first studied in [13] and we refer to [1,5] for books presenting the large literature of the SIR model.

Mean field games study the rational behavior of an infinite number of players. They were introduced recently by Jean-Michel Lasry and Pierre Louis Lions in [16,17,18] and Minyi Huang et al. in [11]. Two important assumptions are made in mean field games: (a) the players are indistinguishable, i.e., one can only observe the number of objects in each state, and (b) as the number of players is infinite, the decisions of an individual player do not affect the dynamics of the

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whole population. These assumptions lead to a simplification of the computation of the equilibrium compared with the computation of the Nash equilibrium of games with a finite number of players, which is known to be a PPAD-complete problem[3] [4]. As a consequence, there has been in the last years a huge literature analyzing mean field games in a wide range of applications in macro-economic models [8], in autonomous vehicles [10], security of communications [14] and traffic modeling [2], to mention a few.

The SIR model and its variations have been also studied from the perspective of mean field games. Some works have analyzed mean field games where the action consists of the vaccinations policies, for instance [15] and its extension with births and deaths [12], and also [7]. In our work we do not consider vaccinations, but the confinement level, or in other words, how players choose to be exposed to the infection. We remark that there are some recent articles that have studied the effect of confinements in the population using mean field games [3,19]. In [3], players can choose a contact rate when they are infected or recovered, while in our work only the susceptible population controls their contact rate. Besides, they consider continuous time and a non-linear cost function, unlike we do. The authors in [19] also consider continuous time and that all players can choose the contact rate with the population. They also consider more variables, such as asymptomatic infected players and the population's age. The main difference between [3,19] and our work, besides the use of discrete time, is that we focus in the end of the epidemic, and that they do not show the existence of the mean field equilibrium nor study its structure. Another related work is [9] where they consider simultaneously the space-time evolution of the epidemics and of the human capital and focus on the benefits of formulating a mean field game.

In this work, we consider that there is an infinite number of players that can decide the probability of being exposed to the infection, i.e., the confinement level. We assume that there is a cost of confinement which is linear and decreasing on the strategy of the players and a cost of being infected per unit time. We also focus on a regime of the end of the epidemic in which the proportion of the infected population decreases over time. In this context, we formulate a mean field game and we show that the solution of this game, i.e., the mean field equilibrium, at the penultimate time step consists of being exposed to the infection with probability one. As a consequence, two strategies are considered in the following: (a) constant, which means that the probability of being exposed to the infection is always one, that is, that rational players are never confined and (b) one jump, which means the players are confined (i.e., not exposed to the infection), at the beginning and, from a given time, the probability of being exposed to the infection is one.

The main contributions of this article are summarized as follows:

- We establish sufficient conditions for the existence of a mean field equilibrium that is constant, i.e., when the mean field equilibrium consists of being

[3] PPAD stands for "polynomial parity arguments on directed graphs". It is a complexity class that is a subclass of NP and is believed to be strictly greater than P.

exposed to the infection with probability one always (see (COND-CONST)). This condition means that, when the cost of infection is small, there exists a mean field equilibrium where players are never confined.
– We establish sufficient conditions for the existence of a mean field equilibrium that is a strategy with one jump (see (COND1-JUMP) and (COND2-JUMP)). This condition means that, when the cost of infection is large, there exists a mean field equilibrium where players are confined at the beginning and, from a given time, are completely exposed to the infection.

Finally, we discuss the numerical experiments we have carried out to analyze the structure of a mean field equilibrium when the aforementioned conditions do not hold. We conclude that when (COND1-JUMP) does not hold the mean field equilibrium with one jump does not seem to exist, but it might exist even though (COND2-JUMP) does not hold.

The rest of the article is organized as follows. In Section 2 we describe the model we study in this article and we formulate the mean field game under analysis. In Section 3 we present some preliminary results regarding the mean-field game. In Section 4 we present our results regarding the existence of a constant mean field equilibrium and in Section 5 we explain our result about the existence of a mean field equilibrium. In Section 6 we discuss the existence of a mean field equilibrium out of the conditions of our main results. In Section 7 we present the main conclusions of our work as well as the future research directions.

2 Model Description

2.1 Notation

We consider a population of homogeneous players that evolve in discrete time from 0 to T. The players are in one of the following three states: susceptible (S), infected (I) or recovered (R). We denote by $m_S(t)$, $m_I(t)$ and $m_R(t)$ the proportion of the population that is in each state.

The dynamics of one player is described as follows. A player encounters other players in a time slot with probability γ. If a player is susceptible and encounters an infected player, then it becomes infected. An infected player recovers in the next time slot with probability ρ. Once a player is recovered, its state does not change. We also consider that a susceptible player can be protected from the infection by choosing the strategy π. A strategy π is a function from $\{0, 1, \dots, T\}$ to $[0, 1]$ and $\pi(t)$ is the probability that a susceptible player at time slot t is exposed to the infection. When $\pi(t) = 0$, the players are confined, or in other words, they are completely protected from the infection at time t; on the other hand, when $\pi(t) = 1$, they are completely exposed to the infection and, therefore, they can be infected if they encounter an infected player. The Markovian behavior of a player is represented in Figure 1.

We are interested in the analysis of this epidemic model with an infinite number of players. In this case, the dynamics of the population is given by the

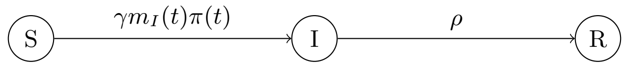

Fig. 1: The dynamics of a player in the epidemic model. An player has three possible states: S (susceptible), I (infected) and R (recovered).

Kolmogorov Equation that takes the following form

$$\begin{cases} m_S(t+1) = m_S(t) - \gamma m_S(t) m_I(t) \pi(t) \\ m_I(t+1) = m_I(t) + \gamma m_S(t) m_I(t) \pi(t) - \rho m_I(t) \\ m_R(t+1) = m_R(t) + \rho m_I(t). \end{cases} \tag{1}$$

Let us make the following assumption.

Assumption 1 (m_I decreasing)**.** *We assume that $m_S(0)\gamma < \rho$.*

We know from (1) that $m_S(t+1) \leq m_S(t)$ for all $t = 0, 1, \ldots, T-1$, i.e., the proportion of the susceptible population is non-increasing with t. Therefore, from the above assumption, it follows that the proportion of infected population decreases with t:

$$\begin{aligned} m_I(t+1) &= m_I(t) + \gamma m_S(t) m_I(t) \pi(t) - \rho m_I(t) \\ &= m_I(t)(1 + \gamma m_S(t)\pi(t) - \rho) \\ &\leq m_I(t)(1 + \gamma m_S(0)\pi(t) - \rho) \\ &\leq m_I(t)(1 + \gamma m_S(0) - \rho) \\ &< m_I(t), \end{aligned}$$

where in the first inequality we use that $m_S(t) \leq m_S(0)$, in the second inequality that $\pi(t) \in [0,1]$ and in the last one the property of Assumption 1. In Section 6, we discuss the difficulties on the analysis of the formulated mean field game when Assumption 1 does not hold.

2.2 Mean Field Game Formulation

We focus on a particular player, that we call Player 0. As we consider a mean field game model, the dynamics of the global population are not affected by Player 0 alone and is driven by Equation (1). Player 0 chooses her confinement strategy π^0, where $\pi^0(t) \in [0,1]$ for all $t = 0, 1, \ldots, T$. We consider that, when $\pi^0(t) = 0$, Player 0 gets confined and therefore, it cannot be infected. But, when $\pi^0(t) = 1$, Player 0 is completely exposed to the infection. The probability that Player 0 is in a given state depends not only on π^0, but also on $m(t)$, the population distribution (which depends on $\pi(t)$).

Let us make the following assumption regarding the cost of a player.

Assumption 2 (Linear confinement cost)**.** *We assume that when a player chooses strategy $\pi^0(t)$, its confinement cost is $c_L - \pi^0(t)$ at time t, where $c_L \geq 1$.*

Therefore the confinement cost of Player 0 is linear with respect to its confinement strategy, and the total cost incurred by the player is the sum of two costs: the confinement cost, which is presented in Assumption 2; and the infection cost, a constant $c_I > 0$ per time unit if the player is infected.

Let $x_i^{\pi^0,\pi}(t)$ be the probability that Player 0 is in state i at time t, where $i \in \{S, I, R\}$. The quantities $x_i^{\pi^0,\pi}(t)$ satisfy the following system of equations:

$$\begin{cases} x_S^{\pi^0,\pi}(t+1) = x_S^{\pi^0,\pi}(t) - \gamma x_S^{\pi^0,\pi}(t) m_I(t) \pi(t) \\ x_I^{\pi^0,\pi}(t+1) = x_I^{\pi^0,\pi}(t) + \gamma x_S^{\pi^0,\pi}(t) m_I(t) \pi(t) - \rho x_I^{\pi^0,\pi}(t) \\ x_R^{\pi^0,\pi}(t+1) = x_R^{\pi^0,\pi}(t) + \rho x_I^{\pi^0,\pi}(t). \end{cases}$$

Note that the above equation is similar to Equation (1) except that it is linear in x whereas Equation (1) is not linear in m.

The expected individual cost of Player 0 is:

$$\sum_{t=0}^{T} \left[x_S^{\pi^0,\pi}(t) f(\pi^0(t)) + c_I x_I^{\pi^0,\pi}(t) \right],$$

where $f(a) = c_L - a$ represents the cost of confinement in a time unit.

We call the *best response to* π, and denote it by $\mathrm{BR}(\pi)$, the set of confinement strategies that minimize the expected cost of Player 0 for a given population strategy π, that is,

$$BR(\pi) = \arg\min_{\pi^0} \sum_{t=0}^{T} \left[x_S^{\pi^0,\pi}(t) f(\pi^0(t)) + c_I x_I^{\pi^0,\pi}(t) \right], \tag{2}$$

which is a non-empty set by compactness of the strategy space.

We define a mean field equilibrium as a fixed point of the best-response function:

Definition 1 (Symmetric Mean Field Equilibrium). *The strategy* π^{MFE} *is a symmetric mean field equilibrium if and only if*

$$\pi^{MFE} \in BR(\pi^{MFE}).$$

This is the classical definition of an equilibrium in a mean field game. The rationale behind this definition is that in a homogeneous population, each player's best-response is the same as that of Player 0. This means that, for a given confinement strategy of the population π, any player of the population chooses the strategy $BR(\pi)$. As in classical games, a mean field equilibrium is a situation where no player has incentive to deviate unilaterally from the selected confinement strategy.

Remark 1. This model is a particular case of the mean field games studied in [6] and therefore, the existence of a mean field equilibrium follows directly. In this work, we go beyond this existence result and our goal is to characterize the structure of the solution of the formulated mean field game.

2.3 Discussion and limits of our assumptions

We have already presented the two main assumptions made in this paper, Assumption 1 and 2, which are not found in most papers on this subject.

Assumption 1 means that we are only studying the end of the epidemic rather than its spread. Therefore, our results focus on strategies for lifting restrictions, rather than on preemptive measures. This assumption makes it easier to find the mean-field equilibrium, and provides more straightforward responses from the players. The generalisation of this work to the whole spread of the epidemic is left for future work.

On the other hand, the main effect Assumption 2 has on our results is the binary choice of players. The linear cost means that when minimizing costs, the best response of players will either be full lockdown or no measures at all. Therefore, instead of having a smooth best response, as in most papers that study similar models, the players' best response will not be continuous. This limitation makes the implementation of a strategy more clear, as the decisions are reduced to two options; but this lack of smoothness makes finding the mean-field equilibrium more difficult.

3 Preliminary Results

We focus on the best-response to π of Player 0. We know that the optimal cost and the best-response verify the following Bellmann equations: for $t = 0, 1, \dots, T-1$,

$$\begin{aligned} V_S^*(t) &= \min_{\pi^0(t)\in[0,1]} \big(f(\pi^0(t)) + (1-\gamma m_I(t)\pi^0(t))V_S^*(t+1) \\ &\qquad +\gamma m_I(t)\pi^0(t)V_I^*(t+1)\big) \\ V_I^*(t) &= c_I + (1-\rho)V_I^*(t+1), \\ BR(\pi)(t) &= \arg\min_{\pi^0(t)\in[0,1]} \big(f(\pi^0(t)) + (1-\gamma m_I(t)\pi^0(t))V_S^*(t+1) \\ &\qquad +\gamma m_I(t)\pi^0(t)V_I^*(t+1)\big), \end{aligned}$$

with $V_S^*(T) = V_I^*(T) = 0$.

Remark 2. The best response of Player 0 for strategy π $BR(\pi)$ is the product of all the best responses at time t $BR(\pi)(t)$ over $t = 0, \dots, T-1$. I.e.

$$BR(\pi) = \{\pi^0 | \pi^0(t) \in BR(\pi)(t), \forall t = 0, \dots, T-1\}$$

We first note that, since $V_S^*(T) = V_I^*(T) = 0$, and from the above Bellmann equations, the best response at time $T-1$ is equal to 1, i.e., the best strategy of Player 0 at the penultimate time step is always one. Therefore, throughout the article, when we say that a strategy (the best-response or the mean field equilibrium) is constant, we mean that it is always one (see Section 4). Likewise, when we say that a strategy has one jump, we mean that there exists a value t_0

such that $0 \leq t_0 < T-1$ and that the considered strategy is zero from 0 to t_0 and one from t_0+1 to T (see Section 5).

We remark that $V_S^*(t)$ is the value for $\pi^0(t)$ that minimizes the following function:

$$c_L - \pi^0(t) + (1-\gamma m_I(t)\pi^0(t))V_S^*(t+1) + \gamma m_I(t)\pi^0(t)V_I^*(t+1).$$

The derivative with respect to $\pi^0(t)$ of the above expression is

$$-1 + \gamma m_I(t)(V_I^*(t+1) - V_S^*(t+1)).$$

This means that the derivative of $V_S^*(t)$ with respect to $\pi^0(t)$ is positive if

$$\gamma m_I(t)\left(V_I^*(t+1) - V_S^*(t+1)\right) > 1, \qquad \text{(COND-BR=0)}$$

in which case the best response at time t is zero, whereas the derivative of $V_S^*(t)$ with respect to $\pi^0(t)$ is negative when

$$\gamma m_I(t)\left(V_I^*(t+1) - V_S^*(t+1)\right) < 1.$$

In case of an equality, the best response can be any value between zero and one. In that case we will consider that Player 0 will decide not to confine, and therefore we have that the best response at time t is one if and only if

$$\gamma m_I(t)\left(V_I^*(t+1) - V_S^*(t+1)\right) \leq 1, \qquad \text{(COND-BR=1)}$$

Hence, Player 0 has to make a binary choice between not confining and confining.

Prior to focus on our analysis of the formulated mean field game, let us present the following result that characterizes the value of $V_I^*(t)$ since it will be useful in the analysis of our work.

Lemma 1. *We have that for* $t = 0, 1, \ldots, T-1$,

$$V_I^*(t) = c_I \sum_{i=0}^{T-1-t} (1-\rho)^i = c_I \frac{1-(1-\rho)^{T-t}}{\rho}.$$

Proof. See Appendix A.

□

Unfortunately, we could not provide a closed-form expression for $V_S^*(t)$ for any t because $V_S^*(t)$ depends on the best-response strategy at every time larger than t. This makes this model extremely difficult to analyze. However, in the following section, we manage to provide sufficient conditions for the existence of a constant mean field equilibrium and, in the next one, of a mean field equilibrium with one jump.

4 Existence of a constant mean field equilibrium

We aim to study the conditions under which there exists a mean field equilibrium that is always one, i.e., a mean field equilibrium π such that $\pi(t) = 1$ for every $t = 0, \ldots, T$. We first show that, if $c_I \leq c_L - 1$, then $V_I^*(t+1)$ is less or equal than $V_S^*(t+1)$ for all t.

Lemma 2. *When $c_I \leq c_L - 1$, we have that $V_S^*(t) \geq V_I^*(t)$ for all $t = 0, 1, \ldots, T$.*

Proof. See Appendix B. □

From this lemma, we have that, when $c_I \leq c_L - 1$, $\gamma m_I(t)\left(V_I^*(t+1) - V_S^*(t+1)\right)$ is always non-positive and, as a consequence, the condition (COND-BR=1) is satisfied for all t. Therefore, we have the following result.

Proposition 1. *When $c_I \leq c_L - 1$, the best response to any π is constant.*

We now focus on the difference between $V_S^*(t)$ and $V_I^*(t)$ and we provide an upper bound of this difference.

Lemma 3. *Assume that the best response to any π at time step t is one. Therefore,*

$$V_S^*(t) - V_I^*(t) < c_I\left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1.$$

Proof. See Appendix C □

From the above result, we now establish a sufficient condition for the best response to any π to be constant.

Proposition 2. *Let $c_I > c_L - 1$. When*

$$\gamma m_I(0)\left(c_I\left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1\right) \leq 1,$$

the best response to any π is constant.

Proof. See Appendix D. □

We now note that

$$\gamma m_I(0)\left(c_I\left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1\right) \leq 1 \iff$$
$$c_I \leq \frac{\rho(1+\gamma m_I(0)(c_L-1))}{\gamma m_I(0)\left(1+\rho-(1-\rho)^{T-1}\right)}.$$

Therefore, according to the above result, the best response to any π is constant when the cost c_I is larger than $c_L - 1$ and less or equal to $\frac{\rho(1+\gamma m_I(0)(c_L-1))}{\gamma m_I(0)(1+\rho-(1-\rho)^{T-1})}$

or, according to Proposition 1, when $c_I \leq c_L - 1$. This means that, under any of these conditions, if we consider that π is constant, the best response to π is constant (according to what we have discussed above), i.e., a constant strategy is a fixed point for the best response function. Thus, according to Definition 1, it follows that a constant strategy is a mean field equilibrium.

Let

$$c_I \leq \max\left(c_L - 1, \frac{\rho(1+\gamma m_I(0)(c_L-1))}{\gamma m_I(0)\left(1+\rho-(1-\rho)^{T-1}\right)}\right) \tag{COND-CONST}$$

We now present the main result of this section, which provides conditions under which a constant mean field equilibrium exists.

Proposition 3. *There exists a mean field equilibrium that is constant when* (COND-CONST) *holds.*

According to this result, we conclude that a mean field equilibrium is constant when the cost of infection is small. This means that no rational player, in this case, will get benefit of changing unilaterally the confinement strategy at any time.

In the next section, we focus on a mean field equilibrium that has one jump. We will thus assume that (COND-CONST) is not satisfied.

5 Existence of a mean field equilibrium with one jump

We first analyze the conditions under which the best response has one jump, i.e., there exists $t_0 < T-1$ such that the best response is

$$\begin{cases} 1 & \text{if } t > t_0 \\ 0 & \text{if } t \leq t_0. \end{cases}$$

We say that a strategy has, at most, one jump when it has one jump or it is constant. Let us now present the following condition that will be required to ensure that the best response has, at most, one jump.

$$c_I \geq \frac{c_L}{(1-\rho)^{T-1}}. \tag{COND1-JUMP}$$

We now show the following result.

Proposition 4. *If* (COND1-JUMP) *holds, then the best response to any π has, at most, one jump.*

Proof. See Appendix E. □

We now focus on the existence of a mean field equilibrium, i.e., we aim to show that there exists a strategy that is a fixed point for the best response function. We consider that $\bar{\pi}$ is the strategy that is a vector with all zeros, i.e.,

$\bar{\pi}(t) = 0$ for all $t = 0, \dots, T$. Let $\widetilde{\pi} \in BR(\bar{\pi})$. When (COND1-JUMP) holds, we know that $\widetilde{\pi}$ has, at most, one jump. In the remainder of this section, when we consider the strategy $\widetilde{\pi}$, we denote by $\widetilde{V}_S^*(t)$ the cost of being susceptible and $\widetilde{m}_S$ and $\widetilde{m}_I$ the proportion of the susceptible and infected population. As we showed in Lemma 1, $V_I^*(t)$ the cost of being infected does not depend on the population's strategy, and therefore when we consider the strategy $\widetilde{\pi}$ the cost of being infected does not change, and we can still denote it as $V_I^*(t)$. We aim to provide conditions such that $\widetilde{\pi} \in BR(\widetilde{\pi})$.

Let t_0 be such that $\widetilde{\pi}(t) = 1$ for all $t > t_0$ and $\widetilde{\pi}(t) = 0$ for all $t \leq t_0$. Since we know that the best response at time $T-1$ is one always, we conclude that t_0 cannot be larger or equal to $T-1$. We assume that $t_0 \geq 0$ (in Remark 4 we deal with the case where this does not occur).

We now show the following result that will be useful to prove the existence of a mean field equilibrium with one jump.

Lemma 4. *Let* (COND1-JUMP)*.*

- *For all* $t \geq 0$, $\widetilde{m}_I(t) \geq m_I(t)$.
- *When* $t > t_0$, $V_I^*(t) \geq V_S^*(t)$
- *When* $t > t_0$, $\widetilde{V}_S^*(t) \geq V_S^*(t)$.

Proof. See Appendix F. □

Using the above results, in the following lemma, we show that the best response to $\widetilde{\pi}$ at time $t_0 + 1$ is one.

Lemma 5. *Let* (COND1-JUMP)*. The best response to* $\widetilde{\pi}$ *at time* $t_0 + 1$ *is one.*

Proof. We know that the best response to $\bar{\pi}$ is one at time $t_0 + 1$. This implies that

$$\gamma m_I(t_0 + 1)(V_I^*(t_0 + 2) - V_S^*(t_0 + 2)) \leq 1. \tag{3}$$

We now remark that, when $t \leq t_0$, $\bar{\pi}(t) = \widetilde{\pi}(t)$ and, as a result, $m_I(t_0 + 1) = \widetilde{m}_I(t_0 + 1)$.

Besides, using Lemma 4, we conclude that $V_I^*(t_0 + 2) - \widetilde{V}_S^*(t_0 + 2)$ is smaller or equal than $V_I^*(t_0 + 2) - V_S^*(t_0 + 2)$. Thus, it follows from (3) that

$$\gamma \widetilde{m}_I(t_0 + 1)(V_I^*(t_0 + 2) - \widetilde{V}_S^*(t_0 + 2)) \leq \gamma m_I(t_0 + 1)(V_I^*(t_0 + 2) - V_S^*(t_0 + 2)) \leq 1,$$

which according to (COND-BR=1) means that the best response to $\widetilde{\pi}$ at time $t_0 + 1$ is one. □

As a consequence of the above reasoning, we have that, when (COND1-JUMP) holds, $\widetilde{\pi}$ is a mean field equilibrium if and only if the best response to $\widetilde{\pi}$ at time t_0 is equal to zero. According to (COND-BR=0), this occurs when

$$\gamma \widetilde{m}_I(t_0)(V_I^*(t_0 + 1) - \widetilde{V}_S^*(t_0 + 1)) > 1.$$

We now notice that for $t \leq t_0$, we have that $\bar{\pi}(t) = \widetilde{\pi}(t)$ and, as a result, we also derive that $m_I(t_0) = \widetilde{m}_I(t_0)$. This implies that the above expression can be alternatively written as follows:

$$\gamma m_I(t_0)(V_I^*(t_0+1) - \widetilde{V}_S^*(t_0+1)) > 1.$$

As a result,

$$\widetilde{\pi} \in BR(\widetilde{\pi}) \iff \gamma m_I(t_0)(V_I^*(t_0+1) - \widetilde{V}_S^*(t_0+1)) > 1.$$

We now aim to investigate the conditions under which the above expression is satisfied. Let us now present the following auxiliary result.

Lemma 6. *Let* (COND1-JUMP)*. For $t > t_0$, $V_I^*(t) - \widetilde{V}_S^*(t)$ is decreasing with t.*

Proof. See Appendix G. □

Taking into account that the best response to any π at time $T-1$ is one and the costs at time T are zero, it follows that $V_I^*(T-1) = c_I$ and $\widetilde{V}_S^*(T-1) = c_L - 1$. Using the result of Lemma 6, we obtain the following result:

Lemma 7. *Let* (COND1-JUMP)*. For all $t > t_0$,*

$$V_I^*(t) - \widetilde{V}_S^*(t) \geq V_I^*(T-1) - \widetilde{V}_S^*(T-1) = c_I - c_L + 1.$$

From this result, we conclude that the condition $\gamma m_I(t_0)(V_I^*(t_0+1) - \widetilde{V}_S^*(t_0+1)) > 1$ is satisfied when

$$\gamma m_I(t_0)(c_I - c_L + 1) > 1. \tag{4}$$

We now remark that, from Assumption 1, $m_I(t_0) > m_I(T)$. Moreover, when (COND1-JUMP) we have that $c_I > c_L - 1$ and, as a result,

$$\gamma m_I(t_0)(c_I - c_L + 1) > \gamma m_I(T)(c_I - c_L + 1).$$

Using (1), we have that $m_I(T) = (1-\rho)^T m_I(0)$. Therefore, (4) is satisfied when

$$\gamma m_I(0)(1-\rho)^T(c_I - c_L + 1) > 1. \tag{COND2-JUMP}$$

From the above reasoning, the next result follows.

Proposition 5. *A mean field equilibrium with one jump exists when* (COND1-JUMP) *and* (COND2-JUMP) *hold.*

In Section 6, we discuss our numerical experiments that show how (COND1-JUMP) and (COND2-JUMP) influence on the existence of a mean field equilibrium.

Let us note that

$$\gamma m_I(0)(1-\rho)^T(c_I - c_L + 1) > 1 \iff c_I > c_L - 1 + \frac{1}{\gamma m_I(0)(1-\rho)^T}.$$

This provides the following expression for the existence of a mean field equilibrium with one jump which is analogous to that of (COND-CONST) for the existence of a mean field equilibrium that is constant:

$$c_I > \max\left(c_L - 1 + \frac{1}{\gamma m_I(0)(1-\rho)^T}, \frac{c_L}{(1-\rho)^{T-1}}\right).$$

According to the derived expression, we conclude that, when c_I is large, there exists a mean field equilibrium that consists of a strategy with one jump. This means that, when players incur a high cost of being infected, they get confined at the beginning of the epidemic and they do not get confined after a fixed threshold time.

Remark 3. We now assume that (COND1-JUMP) and (COND2-JUMP) hold and we consider that the jump is given at $T-2$. According to (COND-BR=0), this occurs when

$$\gamma m_I(T-2)(V_I^*(T-1) - V_S^*(T-1)) > 1.$$

Since $V_I^*(T-1) - V_S^*(T-1) = c_I - c_L + 1$ and $m_I(T-2) = (1-\rho)^{T-2} m_I(0)$ (which holds because $\bar{\pi}$ is a vector with all zeros), the above expression is equivant to

$$\gamma(1-\rho)^{T-2} m_I(0)(c_I - c_L + 1) > 1.$$

This expression is clearly satisfied when (COND2-JUMP) is satisfied. Therefore, we conclude that when (COND1-JUMP) and (COND2-JUMP) hold, the jump is given at time $T-2$.

Remark 4. Let us consider that it does not exist a t_0 such that the best response to $\bar{\pi}$ has one jump. Hence, the best response to $\bar{\pi}$ (which is a vector with all zeros) is constant. According to (COND-BR=1), if (COND1-JUMP) holds this occurs when

$$\gamma m_I(0)(V_I^*(1) - V_S^*(1)) < 1.$$

According the result of Lemma 5, we derive that the best response at time zero is one. As a result, the best response to $\widetilde{\pi}$ is a constant strategy if $\gamma m_I(0)(V_I^*(1) - \widetilde{V}_S^*(1)) < 1$, which implies that a mean field equilibrium that is constant exists when this condition is verified. This provides an additional sufficient condition for the existence of a constant mean field equilibrium to those presented in Section 4.

6 Discussion of (COND1-JUMP) and (COND2-JUMP)

In Proposition 4, we established conditions under which the best response has, at most, one jump. We now aim to analyze the best response when these conditions do not hold and we show that, for this instance, the best response might have multiple jumps.

We consider the following parameters: $T = 100$, $\gamma = 0.85$, $\rho = 0.75$, $c_I = 86$, $c_L = 2$ and $m_S(0) = 0.88$ and $m_I(0) = 0.12$. It is easy to check that these parameters do not satisfy the conditions (COND1-JUMP). In Figure 2, we consider that π is a vector of all ones and we illustrate the best response to π for the considered parameters.

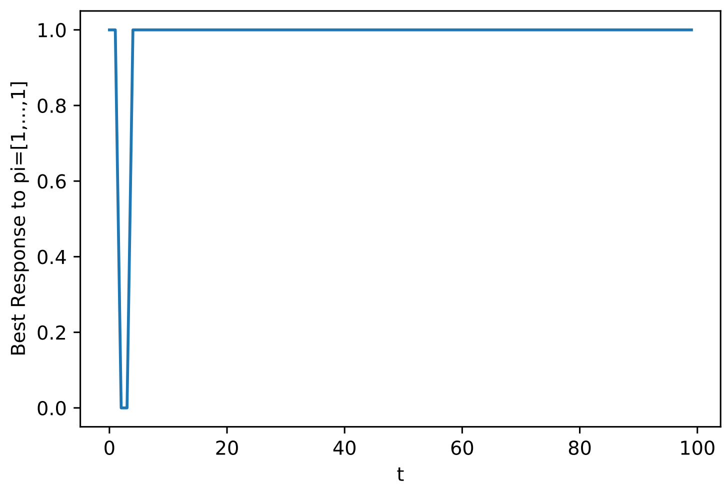

Fig. 2: The best response to $\pi = (1, \ldots, 1)$.

We observe that the best response strategy has two jumps. That is, the behavior of a selfish player consists of being exposed to the infection with probability one at the first two time steps; then, a selfish player would be confined for two time steps and, finally, the best response is equal to one until the end. This result shows that, even though the proportion of infected population decreases with t (and therefore, the maximum number of infected population is achieved at the beginning), when T is large, a selfish player might prefer to be exposed to the infection in the first two time steps instead of being confined. The main reason for this counter-intuitive behavior is that, when the epidemic is long, the player will almost surely get the infection and, therefore, it might decide to be completely exposed in the first time steps.

We also analyzed the structure of a mean field equilibrium with this set of parameters. We first computed the best response to all the strategies π with one jump as well as to the strategy π that is constant. We observed that none of these instances led to a fixed point. Therefore, we conclude that a mean field equilibrium with at most one jump does not always exist. Moreover, we also observed that the fixed point algorithm did not converge to a strategy after 300 iterations. Therefore, when (COND1-JUMP) is not satisfied, the characterization of the mean field equilibrium remains an open question.

We have observed that the fixed point algorithm converges to a mean field equilibrium with at most one jump in all the instances in which (COND1-JUMP) is satisfied, but (COND2-JUMP) not. This numerical work suggests that the mean field equilibrium is constant or has a single jump when (COND1-JUMP) is satisfied. Therefore, (COND1-JUMP) seems to be a necessary and sufficient condition for the existence of a mean field equilibrium with at most one jump.

Finally, we have also studied numerically the structure of a mean field equilibrium when Assumption 1 is not satisfied. For this case, we have seen that the best response might have multiple jumps and the best response algorithm does not always converge. Therefore, the characterization of the mean field equilibrium when Assumption 1 does not hold remains an open question as well.

7 Conclusions and Future Work

We have studied a mean field game in which each player can individually choose how to get confined, i.e., in each time step players can choose the probability of being exposed to the infection. We provide conditions under which there exists (a) a mean field equilibrium that is constant, i.e., it consists of being exposed to the infection with probability one always and (b) a mean field equilibrium that is a strategy with one jump, i.e., it is confined at the beginning and, from a given time, it is completely exposed to the infection.

For future work, we are interested in providing necessary and sufficient conditions for the existence of a mean field equilibrium and also in full characterizing it for the considered assumptions. We would also like to analyze the efficiency of the mean field equilibrium, i.e., if the cost at the mean field equilibrium is much larger than the optimal cost in the system. Finally, we would like to explore this mean field game beyond the ending of the epidemic, for an arbitrary dynamic of the proportion of infected population (i.e., not necessarily decreasing with t).

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A Proof of Lemma 1

We first note that $V_I^*(T-1) = c_I + (1-\rho)V_I^*(T) = c_I$, which clearly verifies the desired condition.

We now assume that $V_I^*(t+1) = c_I \sum_{i=0}^{T-2-t}(1-\rho)^i$, for $t < T-1$ and we verify the following:

$$\begin{aligned} V_I^*(t) &= c_I + (1-\rho)c_I \sum_{i=0}^{T-2-t}(1-\rho)^i = c_I + c_I \sum_{i=1}^{T-1-t}(1-\rho)^i \\ &= c_I \sum_{i=0}^{T-1-t}(1-\rho)^i = c_I \frac{1-(1-\rho)^{T-t}}{\rho}. \end{aligned}$$

And the desired result follows.

B Proof of Lemma 2

Let us first observe that the desired result holds for $t = T$ since $V_S^*(T) = 0$ and $V_I^*(T) = 0$. We now note that, from Lemma 1, we have that $V_I^*(T-1) = c_I$. Besides, using that when $t = T-1$ the best response to π is always one, it follows that $V_S^*(T-1) = c_L - 1$. As a consequence, $V_S^*(T-1) \geq V_I^*(T-1)$ when $c_I \leq c_L - 1$, i.e., the desired result is also satisfied when $t = T-1$. We now assume that $V_S^*(t+1) \geq V_I^*(t+1)$ for $t < T-1$ and we verify that $V_S^*(t) \geq V_I^*(t)$ when $c_I \leq c_L - 1$.

$$\begin{aligned} V_S^*(t) &= \min_{\pi^0(t)\in[0,1]} \left[c_L - \pi^0(t) + (1-\gamma m_I(t)\pi^0(t))V_S^*(t+1) + \gamma m_I(t)\pi^0(t)V_I^*(t+1)\right] \\ &= \min_{\pi^0(t)\in[0,1]} \left[c_L - \pi^0(t) + V_S^*(t+1) + \gamma m_I(t)\pi^0(t)\left(V_I^*(t+1) - V_S^*(t+1)\right)\right] \\ &= c_L - 1 + V_S^*(t+1) + \gamma m_I(t)\left(V_I^*(t+1) - V_S^*(t+1)\right), \end{aligned}$$

where the last equality holds since $V_I^*(t+1) - V_S^*(t+1) \leq 0$ and, therefore, the value that minimizes $c_L - \pi^0(t) + (1-\gamma m_I(t)\pi^0(t))V_S^*(t+1) + \gamma m_I(t)\pi^0(t)V_I^*(t+1)$ is $\pi^0(t) = 1$. Since $V_I^*(t) = c_I + (1-\rho)V_I^*(t+1) < c_I + V_I^*(t+1)$, it follows that

$$\begin{aligned} V_I^*(t) - V_S^*(t) &< c_I + V_I^*(t+1) - \left(c_L - 1 + V_S^*(t+1) + \gamma m_I(t)\left(V_I^*(t+1) - V_S^*(t+1)\right)\right) \\ &= c_I - c_L + 1 + (1-\gamma m_I(t))\left(V_I^*(t+1) - V_S^*(t+1)\right), \end{aligned}$$

which is clearly non-positive because $c_I \leq c_L - 1$ and $\gamma m_I(t) < 1$ and since we assumed that $V_S^*(t+1) \geq V_I^*(t+1)$. And the desired result follows.

C Proof of Lemma 3

Since the best response at time t is one,

$$V_S^*(t) = c_L - 1 + (1-\gamma m_I(t))V_S^*(t+1) + \gamma m_I(t)V_I^*(t+1).$$

Besides,

$$V_I^*(t) = c_I + (1-\rho)V_I^*(t+1) < c_I + V_I^*(t+1).$$

As a result,

$$\begin{aligned} V_I^*(t) - V_S^*(t) &< c_I + V_I^*(t+1) - c_L + 1 - (1-\gamma m_I(t))V_S^*(t+1) - \gamma m_I(t) V_I^*(t+1) \\ &< c_I + V_I^*(t+1) - c_L + 1 - \gamma m_I(t) V_I^*(t+1) \\ &< c_I + V_I^*(t+1) - c_L + 1 \\ &= c_I \left(1 + \frac{1-(1-\rho)^{T-t-1}}{\rho}\right) - c_L + 1 \\ &< c_I \left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1. \end{aligned}$$

And the desired result follows.

D Proof of Proposition 2

We know that the best response to π is one when $t = T - 1$ because the costs at time T are zero. Therefore, we only need to show that the best response to any π is one for all $t < T - 1$.

We assume that the best response to any π is one at time $t + 1$. Therefore, from Lemma 3, it follows that

$$V_S^*(t+1) - V_I^*(t+1) < c_I \left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1.$$

As a result,

$$\gamma m_I(t) \left(V_I^*(t+1) - V_S^*(t+1)\right) < \gamma m_I(t) \left(c_I \left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1\right).$$

From Assumption 1, it follows that $m_I(t) < m_I(0)$ and therefore, the rhs of the above expression is upper bounded by

$$\gamma m_I(0) \left(c_I \left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1\right)$$

because $c_I \left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1 > 0$ since $c_I > c_L - 1$ and $\rho > 0$. Since we have that

$$\gamma m_I(0) \left(c_I \left(1 + \frac{1-(1-\rho)^{T-1}}{\rho}\right) - c_L + 1\right) \leq 1,$$

from the above reasoning, it follows that

$$\gamma m_I(t) \left(V_I^*(t+1) - V_S^*(t+1)\right) < 1,$$

which, according to (COND-BR=1), it implies that the best response to any π at time t is one. And the desired result follows.

E Proof of Proposition 4.

Let us recall that the best response to any π at time $T-1$ is one. We aim to show that, when (COND1-JUMP) holds, the best response to π does not have more than one jump. For a fixed strategy π, let t_0 be the first time (starting from T) such that the best response to π is zero. The desired result follows if we show that for all $t \leq t_0$ the best response to π is zero. Using an induction argument, we assume that, there exists a $\bar{t} \leq t_0$ such that the best response to π at time $\bar{t}$ is zero, which according to (COND-BR=0) is achieved when

$$\gamma m_I(\bar{t})(V_I^*(\bar{t}+1) - V_S^*(\bar{t}+1)) > 1 \tag{5}$$

and we aim to show that

$$\gamma m_I(\bar{t}-1)(V_I^*(\bar{t}) - V_S^*(\bar{t})) > 1 \tag{6}$$

i.e., that the best response to π at time $\bar{t}-1$ is zero as well. Since the best response to π at time $\bar{t}$ is zero, it follows that

$$V_S^*(\bar{t}) = c_L + V_S^*(\bar{t}+1),$$

and we also have that $V_I^*(\bar{t}) = c_I(1-\rho)^{T-\bar{t}-1} + V_I^*(\bar{t}+1)$. As a result,

$$V_I^*(\bar{t}) - V_S^*(\bar{t}) = c_I(1-\rho)^{T-\bar{t}-1} - c_L + V_I^*(\bar{t}+1) - V_S^*(\bar{t}+1).$$

From (5), we obtain that $V_I^*(\bar{t}+1) - V_S^*(\bar{t}+1) > \frac{1}{\gamma m_I(\bar{t})}$, therefore

$$V_I^*(\bar{t}) - V_S^*(\bar{t}) > c_I(1-\rho)^{T-\bar{t}-1} - c_L + \frac{1}{\gamma m_I(\bar{t})}$$

From (COND1-JUMP), we derive that $c_I(1-\rho)^{T-\bar{t}-1} - c_L > 0$, which means that the rhs of the above expression is lower bounded by

$$V_I^*(\bar{t}) - V_S^*(\bar{t}) > \frac{1}{m_I(\bar{t})}.$$

We multiply both sides by $\gamma m_I(\bar{t}-1)$:

$$\gamma m_I(\bar{t}-1)(V_I^*(\bar{t}) - V_S^*(\bar{t})) > \frac{m_I(\bar{t}-1)}{m_I(\bar{t})}.$$

We now notice that $\frac{m_I(\bar{t}-1)}{m_I(\bar{t})} > 1$ due to Assumption 1 and, therefore, (6) holds which implies that the desired result follows.

F Proof of Lemma 4

F.1 $\widetilde{m}_I(t) \geq m_I(t)$

We first show that $\widetilde{m}_I(t) \geq m_I(t)$ for $t \geq 0$. We note that, at time zero, both values coincide, i.e., $\widetilde{m}_I(0) = m_I(0)$. We now assume that for $t \geq 0$, $\widetilde{m}_I(t) \geq$

$m_I(t)$, and we aim to show that $\widetilde{m}_I(t+1) \geq m_I(t+1)$. Using (1) and also that $\bar{\pi}(t)$ is formed by all zeros,

$$m_I(t+1) = m_I(t)(1 + \gamma m_S(t)\bar{\pi}(t) - \rho) = m_I(t)(1-\rho),$$

whereas for $\widetilde{\pi}$ we have

$$\widetilde{m}_I(t+1) = \widetilde{m}_I(t)(1+\gamma\widetilde{m}_S(t)\widetilde{\pi}(t)-\rho) \geq \widetilde{m}_I(t)(1-\rho) \geq m_I(t)(1-\rho) = m_I(t+1).$$

And the desired result follows.

F.2 $V_I^*(t) \geq V_S^*(t)$

We now focus on the proof of $V_I^*(t) \geq V_S^*(t)$ for all $t > t_0$. We first note that $V_I^*(T) = V_S^*(T) = 0$ and, therefore, the desired result follows at time T. We now assume that $V_I^*(t+1) \geq V_S^*(t+1)$ for $t+1 > t_0$ and we aim to show that $V_I^*(t) \geq V_S^*(t)$. Since the best response to $\bar{\pi}$ at time t is one, we have that

$$V_S^*(t) = c_L - 1 + (1 - \gamma m_I(t))V_S^*(t+1) + \gamma m_I(t)V_I^*(t+1)$$

and taking into account that $V_I^*(t) = c_I(1-\rho)^{T-1-t} + V_I^*(t+1)$, we have that

$$V_I^*(t) - V_S^*(t) = c_I(1-\rho)^{T-1-t} - c_L + 1 + (1-\gamma m_I(t))(V_I^*(t+1) - V_S^*(t+1)) \geq 0,$$

which holds since $V_I^*(t+1) \geq V_S^*(t+1)$ and $c_I \geq \frac{c_L}{(1-\rho)^{T-1}} > \frac{c_L}{(1-\rho)^{T-t-1}}$. And the desired result follows.

F.3 $\widetilde{V}_S^*(t) \geq V_S^*(t)$

Finally, we show that $\widetilde{V}_S^*(t) \geq V_S^*(t)$ for all $t > t_0$. We know that $\widetilde{V}_S^*(T) = V_S^*(T) = 0$ since the cost at the end is zero. Therefore, we assume that $\widetilde{V}_S^*(t+1) \geq V_S^*(t+1)$ for $t > t_0$ and we aim to show that $\widetilde{V}_S^*(t) \geq V_S^*(t)$.

We know that the best response to $\bar{\pi}$ for $t > t_0$ is one. Therefore, $V_S^*(t) = c_L - 1 + V_S^*(t+1) + \gamma m_I(t)(V_I^*(t+1) - V_S^*(t+1))$. For $\widetilde{V}_S^*(t)$, we denote by a the best response to $\widetilde{\pi}$ at time t. Thus,

$$\widetilde{V}_S^*(t) = c_L - a + (1 - \gamma\widetilde{m}_I(t)a)\widetilde{V}_S^*(t+1) + \gamma\widetilde{m}_I(t)aV_I^*(t+1).$$

Using that $\widetilde{V}_S^*(t+1) \geq V_S^*(t+1)$ and because $1 - \gamma\widetilde{m}_I(t)a$ is positive, we get

$$\widetilde{V}_S^*(t) \geq c_L - a + (1 - \gamma\widetilde{m}_I(t)a)V_S^*(t+1) + \gamma\widetilde{m}_I(t)aV_I^*(t+1). \tag{7}$$

Therefore, $\widetilde{V}_S^*(t) \geq V_S^*(t)$ is true when

$$\begin{aligned} c_L - a + (1 - \gamma\widetilde{m}_I(t)a)V_S^*(t+1) + \gamma\widetilde{m}_I(t)aV_I^*(t+1) \\ \geq c_L - 1 + V_S^*(t+1) + \gamma m_I(t)(V_I^*(t+1) - V_S^*(t+1)). \end{aligned}$$

Simplifying

$$1-a \geq \gamma(m_I(t)-\widetilde{m}_I(t)a)(V_I^*(t+1)-V_S^*(t+1)).$$

Using that $\widetilde{m}_I(t) \geq m_I(t)$ and since $V_I^*(t+1)-V_S^*(t+1)$ is non-negative, the rhs of the above expression is smaller or equal than $\gamma(1-a)m_I(t)(V_I^*(t+1)-V_S^*(t+1))$. Therefore, a sufficient condition for the desired result to hold is

$$1-a \geq \gamma(1-a)m_I(t)(V_I^*(t+1)-V_S^*(t+1)).$$

We now differentiate two cases: (a) when $a=1$, we have zero in both sides of the expression and therefore, the condition is satisfied; (b) when $a \neq 1$, we divide by $1-a$ both sides of the expression and we get

$$1 \geq \gamma m_I(t)(V_I^*(t+1)-V_S^*(t+1)),$$

which is also satisfied from (COND-BR=1) because the best response to $\bar{\pi}$ at time t is one.

G Proof of Lemma (6)

We aim to show that

$$V_I^*(t)-\widetilde{V}_S^*(t) \geq V_I^*(t+1)-\widetilde{V}_S^*(t+1),$$

for $t>t_0$. From Lemma 5 we know that the best response to $\widetilde{\pi}$ is one for $t>t_0$, because (COND1-JUMP) is satisfied, and therefore,

$$\widetilde{V}_S^*(t)=c_L-1+\widetilde{V}_S^*(t+1)+\gamma m_I(t)(V_I^*(t+1)-\widetilde{V}_S^*(t+1)).$$

From Lemma 1, we have that

$$V_I^*(t)=c_I(1-\rho)^{T-t-1}+V_I^*(t+1).$$

Therefore,

$$V_I^*(t)-\widetilde{V}_S^*(t)=c_I(1-\rho)^{T-t-1}-c_L+1+(1-\gamma m_I(t))(V_I^*(t+1)-\widetilde{V}_S^*(t+1)).$$

Hence, using the above expression, we get that

$$\begin{aligned}V_I^*(t)-\widetilde{V}_S^*(t) \geq V_I^*(t+1)-\widetilde{V}_S^*(t+1) \iff \\ c_I(1-\rho)^{T-t-1}-c_L+1-\gamma m_I(t)(V_I^*(t+1)-\widetilde{V}_S^*(t+1)) \geq 0.\end{aligned}$$

We now note that when $c_I \geq \frac{c_L}{(1-\rho)^{T-1}}$, we get $c_I(1-\rho)^{T-t-1}>c_L$, and as the best response to $\widetilde{\pi}$ is one we have (COND-BR=1). Therefore,

$$\begin{aligned}c_I(1-\rho)^{T-t-1}-c_L+1-\gamma m_I(t)(V_I^*(t+1)-\widetilde{V}_S^*(t+1)) \\ >1-\gamma m_I(t)(V_I^*(t+1)-\widetilde{V}_S^*(t+1)) \geq 0\end{aligned}$$

and the desired result follows.