Energy Packet Networks with Multiple Energy Packet Requirements*

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Abstract

We analyze Energy Packet Networks (EPNs) in which the service centers consist of multiclass queues and the data packets initiate the transfer (i.e., the arrival of a data packet at the battery triggers the movement) with multiple energy packet requirements. In other words, a class-k data packet in cell i is sent successfully to the next cell if there are $c_i^{(k)}$ energy packets and it is dropped otherwise. Besides, we consider that the queues handling data packets operate under one of the following disciplines: First-Input-First-Output (FIFO), Processor Sharing (PS) or Preemptive Last-Input-First-Output (LIFO-PR). This model is an extension of previously studied EPNs [16, 6] where the steady-state distribution of the number of jobs in the queues has a product form. In our model, we show the existence of a product form of the steady-state stationary distribution, where the load of the servers is given by a fixed point expression. We study the existence of a solution to the derived fixed point problem and we provide sufficient conditions for the stability of our model. Finally, we show that, for feed forward EPNs, the load of all the queues can be fully characterized.

1 Introduction

Energy Packet Networks (EPNs) were introduced recently by Gelenbe and his colleagues [11, 12, 13, 16], to model the interactions between energy and IT in a network of computers, routers or sensors. More precisely, it is used to represent the flow of intermittent sources of energy like batteries and solar or wind based generators and their interactions with IT devices consuming energy like sensors, cpu, storage systems and networking elements.

The key idea of EPNs is to represent energy with packets of discrete units called Energy Packets (EPs). Each EP models a certain number of Joules. Since the EPs are produced by an intermittent source of energy (typically solar and wind), the flow of EPs is associated with some random processes. EPs are consumed by some devices after some random duration to perform requested works or can also be stored in a battery from which they can also leak after a random delay.

In the one of the first EPN model presented for instance in [16], one represents the energy as EPs and the workload as Data Packets (DPs). To transmit a DP between two cells, one must use one EP. Hence, each cell in the network is associated with a server queue to store the DPs and a battery (the EP queue) to keep the energy. In this paper (i.e. [16]), the EPs are sent to the DP queue and triggers the customer movement between workload queues in the network. When

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an EP arrives at a DP queue which is not backlogged, the energy is lost. This model have been generalized in several directions:

- One EP may be not sufficient to send a DP [14]. In [21] the number of EP needed to transfer a DP is a constant K. This energy can even be a discrete random variable like in [20] or associated with a continuous process in [1].
- In [22], the DP queue is the initiator of the transfer. The arrival of a DP at the battery triggers the deletion of a an EP and the movement of the data. If a data packet does not find an energy packet, it is lost. Thus we represent losses of DP while the initial model is used to represent packets delayed due to the lack of energy.

Some of these EPN models (not all of them) are G-networks of queues and therefore, they are related to the models developed after the seminal papers by Gelenbe on networks of queues with positive and negative customers [8], queues with triggers [9], or queues with batch deletions [10]. G-networks are also known as networks of queues and signals to put more emphasis on the new synchronized transitions provoked by the signals [3]. The importance of these models resides on the existence of a product form of the steady-state distribution of jobs in the queues since that allows to optimize the system for some utility function like losses or response time [16, 17, 4]. Typically, due to this closed form solution, EPN models allow to optimize the energy distribution [20] and design networks with energy harvesting [15].

Note however that EPN models are not always related to G-networks. For instance, the model in [13] is based on queues without signals while in [1] the authors use a diffusion approximation to solve the interactions between IT and energy. They describe the system with a 4 states model to represent the interactions between the energy harvesting and the data censoring and communication. In [18] and [22], the number of EP and DP are not explicitly represented in the model but the evolution of the difference between them (i.e. $DP - EP$) is studied under classical assumptions on the arrivals and the interactions. Thus, the stochastic process describing the population has a support on Z, a non conventional feature in queueing theory.

An independent approach to the EPNs has been presented in the electrical engineering literature under the name "power packet", see [25, 24] and references therein. In their approach, power packets are modeled as a pulse of current and are associated with a header and a protocol to control the routing using some hardware switching.

In this article, we consider multiple classes of DPs, where a class of a DP determines the number of EPs required to be sent and the routing matrix. We assume as in [22] that the DP queue is the initiator of the transfer, that is, the arrival of a DP at the battery triggers the movement and, if a data packet does not find enough energy packets, it is lost. As usual with queues with multiple classes of customers, we consider several queuing disciplines. Here as in [6] we consider that the DP queues operate under one of the following three types: the FIFO, LIFO-PR and PS queues. This disciplines have already been considered for BCMP queues.

We show that the steady-state distribution of jobs in the queues has a product form provided that a stable solution of a fixed point problem exists. Then, assuming that the underlying queuing network is ergodic, we show that if a stable solution of a fixed problem exists, then it is unique. Moreover, we provide sufficient conditions for the stability of the EPN with an arbitrary topology. Finally, we focus on feed forward networks and we show that the loads of all the queues can be characterized following the topological order of the network. Indeed, we observe that the fixed point problem is reduced to compute the roots of a polynomial.

The remainder of the article is organized as follows. In Section 2, we describe the model. The main results of the article are given in Section 3. In Section 4, we focus on networks without

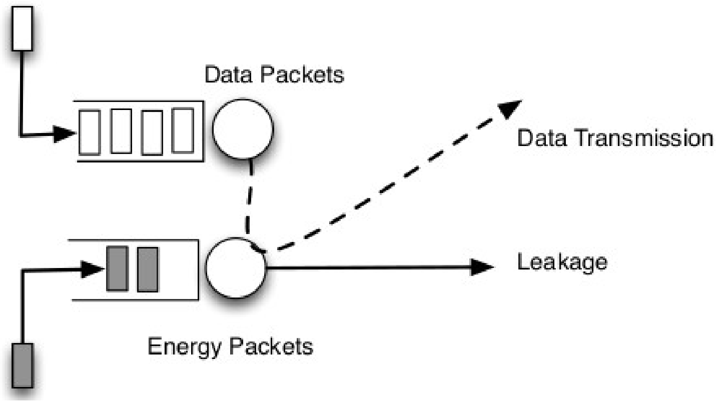

Figure 1: A cell of the EPN model under study.

directed cycles. Finally, we present the conclusions of this work in Section 5.

2 Model Description

We study a EPN with N cells in an open network, where each of the cells is formed by one server that stores DPs and one battery that stores EPs. We consider that there are K classes of DPs.

DPs arrive to each cell from outside the network following a Poisson distribution and we denote by $\lambda_i^{(k)}$ the arrival rate of class-k DPs to cell i. Likewise, EPs arrive to cell i following a Poisson distribution with rate α_i. We consider that there are energy leakages. This means that there are losses of EPs with exponential times. Let $\beta_i > 0$ be the leakage rate of one EP in the battery of cell i. The service time of DPs is assumed to be exponentially distributed. We denote by $\mu_i^{(k)} > 0$ the rate of the service time of class k jobs in the DP queue of cell i.

We consider a EPN where the DPs initiates the transfer. This means that, upon service in cell i, a DP of class k is sent to the battery of the same cell. The transfer is successful if there are at least $c_i^{(k)}$ EPs, in which case the DP is routed to the next cell and the $c_i^{(k)}$ EPs disappear. If the data packet finds less than $c_i^{(k)}$ EPs, then it is dropped. Furthermore we assume that there is some energy consumption to send the DP even if it fails. If there is a lack of energy before the end of the DP emission, the DP is lost and the energy consumed for the tentative is also lost. Thus, the battery gets empty.

Furthermore we assume that some nodes are on the boundary of the network and send DP to applications on the same machines or terminals. We assume that DPs do not consume energy when they leave the network because they stay on the same terminal but on the application level. The DPs move from one cell to another according to a fixed probability matrix, where $P_{i,j}^{(k)}$ represents the probability for a class k DP to move from the DP queue of cell i to the DP queue of cell j in case of a successful transfer, i.e., if there are more than $c_i^{(k)}$ EPs in the battery of cell i. Upon service in cell i, a DP packet of class k leaves the system with probability $d_i^{(k)}$. Hence, for all k and i, it follows that

$$d_i^{(k)} + \sum_{j=1}^{N} P_{i,j}^{(k)} = 1.$$

Like in [6], we consider three server types: First-Input-First-Output (FIFO), Processor Sharing

(PS) and Preemptive Last-Input-First-Output (LIFO-PR). The FIFO discipline consists of giving serving to jobs in order of arrival. In the PS discipline all the jobs get a proportion of the processing capacity of the server and, if the number of jobs of class k jobs present in the system is x_k, the rate at which DPs of class k are served is given by $\frac{x_k}{\sum_{j=1}^{K} x_j}$. In the LIFO-PR discipline, the customers in the system constitute a stack, and the system serves always the customer that has been waiting for the shortest time.

2.1 State representation

We shall denote the state at time t of the queuing network by the vector $(\vec{X}, \vec{Z})$ where $\vec{X} = (x_1, ..., x_N)$ and x_i represents the state of DP service center of cell i and $\vec{Z} = (z_1, ..., z_N)$ and z_i is the number of EPs in the battery of cell i. The vector x_i depends on the queuing discipline of the DP queue of cell i and $||x_i||$ will be the total number of DPs in that queue.

For FIFO and LIFO-PR servers, the instantaneous value of the state x_i of server of cell i is represented by the vector $(r_{i,j})$ whose length is the number of customers in the queue and whose j-th element is the class index of the j-th customer in the queue. Furthermore, the customers are ordered according to the service order and it is always the customer at the head of the list which is in service. We denote by $r_{i,1}$ the class number of the customer in service and by $r_{i,\infty}$ the class number of the last customer in the queue.

For the PS servers, the instantaneous value of the state x_i is represented by the vector $(x_{i,k})$, where the k-th element represents the number of customers of class k at queue i.

3 Main Results

Let $\Pi(\vec{X}, \vec{Z})$ denote the stationary probability distribution of the state of the network, if it exists. The following result establishes the product form solution of the network being considered.

Theorem 1. *Consider the EPN previously defined. If the system of non-linear equations:*

$$\rho_i^{(k)} = \frac{\lambda_i^{(k)} + \sum_{j=1}^{N} \mu_j^{(k)} \rho_j^{(k)} P_{j,i}^{(k)} (\gamma_j)^{c_j^{(k)}}}{\mu_i^{(k)}}, \tag{1}$$

and

$$\gamma_i = \frac{\alpha_i}{\beta_i + \sum_{l=1}^{K} \rho_i^{(l)} \mu_i^{(l)} \sum_{m=0}^{c_i^{(l)}-1} (\gamma_i)^m \sum_{j=1}^{N} P_{i,j}^{(l)}} \tag{2}$$

has a solution such that: for each pair i, k, $0 < \rho_i^{(k)}$ and for each DP queue i, $\sum_{k=1}^{K} \rho_i^{(k)} < 1$ and EP queue i, $\gamma_i < 1$, then the stationary distribution of the network state is:

$$\Pi(\vec{X}, \vec{Z}) = G \prod_{i=1}^{N} g_i(x_i) \prod_{i=1}^{N} (1 - \gamma_i)(\gamma_i)^{z_i}, \tag{3}$$

where each $g_i(x_i)$ depends on the discipline of the service center of cell i. The $g_i(x_i)$ in Eq. 3 have the following form:

FIFO *If the service center is FIFO, then*

$$g_i(x_i) = \prod_{n=1}^{||x_i||} [\rho_i^{(r_{i,n})}] \tag{4}$$

PS *If the service center is PS, then*

$$g_i(x_i) = ||x_i||! \prod_{k=1}^{K} \frac{(\rho_i^{(k)})^{x_{i,k}}}{x_{i,k}!} \tag{5}$$

LIFO-PR *If the service center is LIFO-PR, then*

$$g_i(x_i) = \prod_{n=1}^{||x_i||} [\rho_i^{(r_{i,n})}] \tag{6}$$

and G is the normalization constant. Since the network is open, G has a closed form expression which is given in Theorem 2.

The proof is based on simple algebraic manipulations of global balance equations, since it is not possible to use the "local balance" equations for customer classes at servers.

In order to carry out algebraic manipulations of the stationary Chapman-Kolmogorov (global balance) equations, we introduce some notation and develop intermediate results:

- The state dependent service rate for customers at the service center of cell j will be denoted by $M_j^{(l)}(x_j)$ where x_j refers to the state of the service center and l is the class of the customer concerned. From the definition of the service rate $\mu_j^{(l)}$, we obtain for the types of servers:

  FIFO : $M_j^{(l)}(x_j) = 1_{\{||x_j||>0\}}\mu_j 1_{\{r_{j,1}=l\}}$,

  LIFO-PR: $M_j^{(l)}(x_j) = 1_{\{||x_j||>0\}}\mu_j^{(l)} 1_{\{r_{j,1}=l\}}$,

  PS: $M_j^{(l)}(x_j) = \mu_j^{(l)} \frac{x_{j,l}}{||x_j||} 1_{\{||x_j||>0\}}$.

- $E(x_j \ominus e^{(l)})$ is the condition which establishes that it is possible to reach state x_j by an arrival of a DP customer of class l

  FIFO: $E(x_j \ominus e^{(l)}) = 1_{\{r_{j,\infty}=l\}}$,

  LIFO-PR $E(x_j \ominus e^{(l)}) = 1_{\{r_{j,1}=l\}}$,

  PS $E(x_j \ominus e^{(l)}) = 1_{\{x_{j,l}>0\}}$.

- Denote by $(x_j \oplus e^{(l)})$ the state of the server of cell j obtained by *adding* a DP of class l. This operation must follow the service discipline.

- Denote by $(\vec{X} \boxplus e_j^{(l)})$ the vector of DP queue states $\vec{X} = (x_1, x_2, \ldots, x_N)$ where the state of DP queue j, x_j, is replaced by $(x_j \oplus e^{(l)})$.

- Denote by $(\vec{Z} + e_i)$ the states of the batteries with one more EP in the battery of cell i

- Denote by $(x_i \ominus e^{(k)})$ the state whose successor is x_i when an arrival of a class k customer occurs (if it exists, otherwise $(x_i \ominus e_i^{(k)})$ is not defined).

- Denote by $(\vec{X} \boxminus e_j^{(l)})$ the vector of DP queue states $\vec{X} = (x_1, x_2, \ldots, x_N)$ where the state of queue j, x_j, is replaced by $(x_j \ominus e^{(l)})$.

We now provide the following technical lemmas, whose proofs can be found in Appendix A.

Lemma 1. *Let j be an arbitrary queue, for all the disciplines we consider for that queue we have:*

$$\sum_{l=1}^{K} M_j^{(l)}(\vec{X} \oplus e_j^{(l)}) \frac{g_j(x_j \oplus e^{(l)})}{g_j(x_j)} = \sum_{l=1}^{K} \mu_j^{(l)} \rho_j^{(l)} \tag{7}$$

Lemma 2. *For all the disciplines we consider (for queue j) we have the following relation, sometimes called a station balance:*

$$\sum_{l=1}^{K} M_j^{(l)}(x_j) = \sum_{l=1}^{K} \frac{g_j(x_j \ominus e^{(l)})}{g_j(x_j)} E(\vec{X} \ominus e_j^{(l)}) \left[\lambda_j^{(l)} + \sum_{i=1}^{N} \mu_i^{(l)} \rho_i^{(l)} (\gamma_i)^{c_i^{(l)}} P_{i,j}^{(l)} \right] \tag{8}$$

Lemma 3. *The following flow equation is satisfied:*

$$\sum_{j=1}^{N}\sum_{l=1}^{K} \lambda_j^{(l)} + \sum_{j=1}^{N} \alpha_j = \sum_{j=1}^{N} \beta_j \gamma_j + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1} \mu_j^{(l)} \rho_j^{(l)} (\gamma_j)^m P_{j,i}^{(l)} + \sum_{j=1}^{N}\sum_{l=1}^{K} \mu_j^{(l)} \rho_j^{(l)} d_j^{(l)}. \tag{9}$$

Let us now turn back to the proof of the Theorem 1. Consider the global balance equation:

$$\begin{aligned}
&\Pi(\vec{X},\vec{Z})[\sum_{j=1}^{N}\sum_{l=1}^{K}(\lambda_j^{(l)} + M_j^{(l)}(x_j)) + \sum_{j=1}^{N} \alpha_j + \sum_{j=1}^{N} \beta_j 1_{\{z_j>0\}}] \\
= \quad & \sum_{j=1}^{N} \Pi(\vec{X}, \vec{Z} - e_j) \alpha_j 1_{\{z_j>0\}} \\
+ \quad & \sum_{j=1}^{N} \Pi(\vec{X}, \vec{Z} + e_j) \beta_j \\
+ \quad & \sum_{j=1}^{N}\sum_{l=1}^{K} \Pi(\vec{X} \boxminus e_j^{(l)}, \vec{Z}) \lambda_j^{(l)} E(x_j \ominus e^{(l)}) \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{K} M_j^{(l)}(x_j \oplus e^{(l)}) \Pi(\vec{X} \boxplus e_j^{(l)} \boxminus e_i^{(l)}, \vec{Z} + c_j^{(l)} e_j) P_{j,i}^{(l)} E(x_i \ominus e^{(l)}) \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1} M_j^{(l)}(x_j \oplus e^{(l)}) \Pi(\vec{X} \boxplus e_j^{(l)}, \vec{Z} + m e_j) P_{j,i}^{(l)} 1_{\{z_j=0\}} \\
+ \quad & \sum_{j=1}^{N}\sum_{l=1}^{K} M_j^{(l)}(x_j \oplus e^{(l)}) \Pi(\vec{X} \boxplus e_j^{(l)}, \vec{Z}) d_j^{(l)}.
\end{aligned}$$

We divide both sides by $\Pi(\vec{X},\vec{Z})$ and we assume that there is a product form solution.

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{K}(\lambda_j^{(l)}+M_j^{(l)}(x_j))+\sum_{j=1}^{N}\alpha_j+\sum_{j=1}^{N}\beta_j 1_{\{z_j>0\}}\\
&= \sum_{j=1}^{N}\frac{\alpha_j}{\gamma_j}1_{\{z_j>0\}}\\
&+ \sum_{j=1}^{N}\beta_j\gamma_j\\
&+ \sum_{j=1}^{N}\sum_{l=1}^{K}\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}\lambda_j^{(l)}E(x_j\ominus e^{(l)})\\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{K}M_j^{(l)}(x_j\oplus e^{(l)})\frac{g_j(x_j\oplus e^{(l)})}{g_j(x_j)}\frac{g_i(x_i\ominus e^{(l)})}{g_i(x_i)}(\gamma_j)^{c_j^{(l)}}P_{j,i}^{(l)}E(x_i\ominus e^{(l)})\\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1}M_j^{(l)}(x_j\oplus e^{(l)})\frac{g_j(x_j\oplus e^{(l)})}{g_j(x_j)}(\gamma_j)^m P_{j,i}^{(l)}1_{\{z_j=0\}}\\
&+ \sum_{j=1}^{N}\sum_{l=1}^{K}M_j^{(l)}(x_j\oplus e^{(l)})\frac{g_j(x_j\oplus e^{(l)})}{g_j(x_j)}d_j^{(l)}.
\end{aligned}$$

According to Lemma 1, we have $\sum_{l=1}^{K}M_j^{(l)}(x_j\oplus e^{(l)})\frac{g_j(x_j\oplus e^{(l)})}{g_j(x_j)}$ is equal to $\sum_{l=1}^{K}\mu_j^{(l)}\rho_j^{(l)}$ After some substitution, we obtain:

$$\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{K}(\lambda_j^{(l)}+M_j^{(l)}(x_j))+\sum_{j=1}^{N}\alpha_j+\sum_{j=1}^{N}\beta_j 1_{\{z_j>0\}}\\
&= \sum_{j=1}^{N}\frac{\alpha_j}{\gamma_j}1_{\{z_j>0\}}\\
&+ \sum_{j=1}^{N}\beta_j\gamma_j\\
&+ \sum_{j=1}^{N}\sum_{l=1}^{K}\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}\lambda_j^{(l)}E(x_j\ominus e^{(l)})\\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{K}\mu_j^{(l)}\rho_j^{(l)}\frac{g_i(x_i\ominus e^{(l)})}{g_i(x_i)}(\gamma_j)^{c_j^{(l)}}P_{j,i}^{(l)}E(x_i\ominus e^{(l)})\\
&+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1}\mu_j^{(l)}\rho_j^{(l)}(\gamma_j)^m P_{j,i}^{(l)}1_{\{z_j=0\}}\\
&+ \sum_{j=1}^{N}\sum_{l=1}^{K}\mu_j^{(l)}\rho_j^{(l)}d_j^{(l)}.
\end{aligned}$$

We write $1_{\{z_j=0\}} = 1 - 1_{\{z_j>0\}}$ and we move the negative term on the left hand side.

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{K}(\lambda_j^{(l)} + M_j^{(l)}(x_j)) + \sum_{j=1}^{N}\alpha_j + \sum_{j=1}^{N}\beta_j 1_{\{z_j>0\}} \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1} \mu_j^{(l)}\rho_j^{(l)}(\gamma_j)^m P_{j,i}^{(l)} 1_{\{z_j>0\}} \\
= \quad & \sum_{j=1}^{N}\frac{\alpha_j}{\gamma_j}1_{\{z_j>0\}} \\
+ \quad & \sum_{j=1}^{N}\beta_j\gamma_j \\
+ \quad & \sum_{j=1}^{N}\sum_{l=1}^{K}\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}\lambda_j^{(l)}E(x_j\ominus e^{(l)}) \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{K}\mu_j^{(l)}\rho_j^{(l)}\frac{g_i(x_i\ominus e^{(l)})}{g_i(x_i)}(\gamma_j)^{c_j^{(l)}}P_{j,i}^{(l)}E(x_i\ominus e^{(l)}) \\
+ \quad & \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1} \mu_j^{(l)}\rho_j^{(l)}(\gamma_j)^m P_{j,i}^{(l)} \\
+ \quad & \sum_{j=1}^{N}\sum_{l=1}^{K}\mu_j^{(l)}\rho_j^{(l)}d_j^{(l)}.
\end{aligned}
$$

The flow equation on γ_i, which is given in Eq. 2, makes the terms with step function $1_{\{z_j>0\}}$ cancel. Therefore, after reordering the terms, substituting i and j of the r.h.s. and moving all the terms with $\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}$ in the l.h.s where they are factorized :

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{l=1}^{K}\lambda_j^{(l)} + \sum_{j=1}^{N}\alpha_j + \sum_{j=1}^{N}\left[\sum_{l=1}^{K}M_j^{(l)}(x_j) - \sum_{l=1}^{K}\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}\lambda_j^{(l)}E(x_j\ominus e^{(l)})\right. \\
- \quad & \left.\sum_{i=1}^{N}\sum_{l=1}^{K}\mu_i^{(l)}\rho_i^{(l)}\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}(\gamma_i)^{c_i^{(l)}}P_{i,j}E(x_j\ominus e^{(l)})\right] \\
= \quad & \sum_{j=1}^{N}\beta_j\gamma_j + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1}\mu_j^{(l)}\rho_j^{(l)}(\gamma_j)^m P_{j,i}^{(l)} \\
+ \quad & \sum_{j=1}^{N}\sum_{l=1}^{K}\mu_j^{(l)}\rho_j^{(l)}d_j^{(l)}.
\end{aligned}
$$

According to Lemma 2, for all service discipline we have:

$$
\sum_{l=1}^{K}M_j^{(l)}(x_j) = \sum_{l=1}^{K}\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}\lambda_j^{(l)}E(x_j\ominus e^{(l)}) + \sum_{i=1}^{N}\sum_{l=1}^{K}\mu_i^{(l)}\rho_i^{(l)}\frac{g_j(x_j\ominus e^{(l)})}{g_j(x_j)}(\gamma_i)^{c_i^{(l)}}P_{j,i}^{(l)}E(x_i\ominus e^{(l)})
$$

Therefore the terms in the l.h.s. cancel and we get:

$$\sum_{j=1}^{N}\sum_{l=1}^{K}\lambda_j^{(l)} + \sum_{j=1}^{N}\alpha_j = \sum_{j=1}^{N}\beta_j\gamma_j + \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{K}\sum_{m=0}^{c_j^{(l)}-1}\mu_j^{(l)}\rho_j^{(l)}(\gamma_j)^m P_{j,i}^{(l)} + \sum_{j=1}^{N}\sum_{l=1}^{K}\mu_j^{(l)}\rho_j^{(l)}d_j^{(l)}. \quad (10)$$

Finally, Lemma 3 shows that this flow equation is satisfied. This concludes the proof.

□

As in BCMP [2] theorem, we can also compute the steady state distribution of the number of customers of each class in each queue. Let y_i be the vector whose elements are $(y_{i,k})$ the number of customers of class k in the server of cell i. Let $\vec{Y}$ be the vector of vectors (y_i).

Theorem 2. *If the system of equations (1), and (2) has a solution then, the steady state distribution of the DPs* $\Pi(\vec{Y})$ *is given by*

$$\Pi(\vec{Y}) = \prod_{i=1}^{N} h_i(y_i) \quad (11)$$

where the marginal probabilities $h_i(y_i)$ *have the following form :*

$$h_i(y_i) = (1 - \sum_{k=1}^{K}\rho_i^{(k)})|y_i|! \prod_{k=1}^{K}[(\rho_i^{(k)})^{y_{i,k}}/y_{i,k}!] \quad (12)$$

Proof. First one can easily check that:

$$\sum_{\vec{Z}}\sum_{i}(1-\gamma_i)(\gamma_i)^{z_i} = 1.$$

Therefore

$$\Pi(\vec{X}) = \sum_{\vec{Z}}\Pi(\vec{X},\vec{Z}) = G \quad \prod_{i=1}^{N} g_i(x_i). \quad (13)$$

Let i be an arbitrary DP queue, let s be the function which computes for an arbitrary state x_i the vector y_i. Let $R_i(y_i)$ be the set of states for a DP queue such that $s(x_i) = y_i$. By construction we have:

$$\pi(y_i) = \sum_{\vec{X}/x_i \in R_i(y_i)} \Pi(\vec{X})$$

As the solution is separable in the components of vector $\vec{X}$, we obtain:

$$\pi(y_i) = \sum_{x_i \in R_i(y_i)} g_i(x_i)$$

From Eq. 1, Eq. 2 and Eq. 3 it is clear that if x_i and x_i' are such that $s(x_i) = s(x_i')$ then $g_i(x_i) = g_i(x_i')$. Therefore

$$\pi(y_i) = Card(R_i(y_i))g_i(x_i),$$

for an arbitrary x_i in $R_i(y_i)$, where $Card()$ is the cardinal of a set. For PS queues, we have $y_i = x_i$ and $Card(R_i(y_i)) = 1$. For a FIFO or LIFO-PR queue, the cardinal of the set is a multinomial coefficient

$$Card(R_i(y_i)) = \frac{||y_i||!}{\prod_{k=1}^{K}(y_{i,k}!)}.$$

Thus Eq. 12 is proved. Consider now $\Pi(\vec{Y})$.

$$\Pi(\vec{Y}) = \sum_{\vec{X}/\forall i, x_i \in R_i(y_i)} \Pi(\vec{X})$$

As the solution is multiplicative, we have:

$$\Pi(\vec{Y}) = G \prod_{i=1}^{N} \sum_{x_i \in R_i(y_i)} g_i(x_i).$$

Therefore

$$\Pi(\vec{Y}) = G \prod_{i=1}^{N} ||y_i||! \prod_{k=1}^{K} \frac{(\rho_i^{(k)})^{y_{i,k}}}{y_{i,k}!}.$$

We now compute the value of G using that $\sum_{\vec{X},\vec{Z}} \pi(\vec{X},\vec{Z}) = 1$. By Eq. 13 and the definition of $\vec{Y}$, we get that

$$\sum_{\vec{X},\vec{Z}} \pi(\vec{X},\vec{Z}) = \sum_{\vec{X}} \pi(\vec{X}) = \sum_{\vec{Y}} \Pi(\vec{Y}) = 1.$$

Therefore,

$$\sum_{\vec{Y}} G \prod_{i=1}^{N} ||y_i||! \prod_{k=1}^{K} \frac{(\rho_i^{(k)})^{y_{i,k}}}{y_{i,k}!} = 1,$$

which is equivalent to

$$G \prod_{i=1}^{N} \sum_{y_i=0}^{\infty} ||y_i||! \prod_{k=1}^{K} \frac{(\rho_i^{(k)})^{y_{i,k}}}{y_{i,k}!} = 1,$$

By the multinomial theorem, it follows that

$$G \prod_{i=1}^{N} \sum_{y_i=0}^{\infty} \left(\sum_{k=1}^{K} \rho_i^{(k)} \right)^{y_i} = 1.$$

Since by the stability condition $\sum_{k=1}^{K} \rho_i^{(k)} < 1$, it follows that

$$G \prod_{i=1}^{N} \frac{1}{1 - \sum_{k=1}^{K} \rho_i^{(k)}} = 1.$$

This proves that the normalization constant is $G = \prod_{i=1}^{N}(1 - \sum_{k=1}^{K} \rho_i^{(k)})$ and the desired result follows. $\square$

3.1 Existence of a fixed point solution

The main result of our work states that a product form of the EPN exists if the solution of the fixed point problem defined in Eq. 1 and Eq. 2 is stable. In this section, we study the existence of a solution to this fixed point problem, whereas the question of the stability is addressed next.

Let $\vec{\rho}_i = (\rho_i^{(1)}, \rho_i^{(2)}, \ldots, \rho_i^{(K)})$, for $i = 1, \ldots, N$ and $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_N)$. We define the following function $F(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_N, \gamma)$ whose image is a $N \times (K+1)$ matrix such that $F_{i,j}(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_N, \gamma)$, for $i = 1, \ldots, N$ and $j = 0, 1, \ldots, K$, where for $i = 1, \ldots, N$

$$F_{i0}(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_N, \gamma) = \frac{\alpha_i}{\beta_i + \sum_{l=1}^{K} \rho_i^{(l)} \mu_i^{(l)} \sum_{m=0}^{c_i^{(l)}-1} (\gamma_i)^m \sum_{j=1}^{N} P_{i,j}^{(l)}}.$$

and for $i = 1, \ldots, N$ and $j = 1, \ldots, K$,

$$F_{ij}(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_N, \gamma) = \frac{\lambda_i^{(k)} + \sum_{j=1}^N \mu_j^{(k)} \rho_j^{(k)} P_{j,i}^{(k)} (\gamma_j)^{c_j^{(k)}}}{\mu_i^{(k)}}.$$

We aim to study the existence of a fixed point for the function F. We first define a function H such that the values of its image are the minimum between the values of the image of F and one, that is, $H_{i,j}(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_N, \gamma) = \min(1, F_{i,j}(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_N, \gamma))$ for $i = 1, \ldots, N$ and $j = 0, 1, \ldots, K$.

We now show that the function H satisfies the conditions of the Brouwer's fixed point theorem.

Lemma 4. *The function H has a fixed point in $[0,1]^{N(K+1)}$.*

Proof. We first note that function H is continuous since $\beta_i > 0$ for all cell i and $\mu_i^{(k)} > 0$ for all i and k. By definition, the image of the function H is contained in $[0,1]^{N(K+1)}$. As a result, since $[0,1]^{N(K+1)}$ is compact and non-empty, the conditions of the Brouwer's fixed point theorem are satisfied and, therefore, the function H has a fixed point. □

We now study the relation between the fixed points of H and of F. In the following result, we show that there are fixed points of H that are also fixed points of F.

Lemma 5. *Let $\vec{x} = H(\vec{x})$ be a fixed point of H. If all the components of $\vec{x}$ are less than one, then $\vec{x}$ is also a fixed point of F.*

Proof. It follows immediately from the definition of the function H. □

We now define a stable fixed point as the solution of the fixed point problem defined in Eq. 1 and Eq. 2 satisfying that $\gamma_i < 1$ and $\sum_{k=1}^K \rho_i^{(k)} < 1$ for all i. From the previous results, we conclude that, assuming erdocity, if H has a stable fixed point then it is unique as the steady-state distribution of an ergodic Markov Chain is unique.

Proposition 1. *Assume ergocity. If there exists a stable fixed point, then it is unique.*

We have analyzed the fixed point of the function H by performing simulations of a large amount of simulations with a huge range of values of the parameters. We have observed that the iterations of the fixed point algorithm applied to H converged to a unique value for all the cases. Unfortunately, we have not succeed in proving the convergence of the iterations of the fixed point algorithm and therefore it remains as an open question.

3.2 Stability

We now investigate the stability of the EPNs we study in this work. We know that the EPN is stable if and only if $\gamma_i < 1$ and $\sum_{k=1}^K \rho_i^{(k)} < 1$ for all cell i. Due to the lack of an explicit expression of the values of loads of EPs and DPs, providing necessary and sufficient conditions for the stability of this model seems to be an impossible task. However, in this section, we succeed in showing a sufficient condition for the stability of the EPN under consideration.

We first define the reduced network as the Jackson network that is formed by the servers only, i.e., without considering the batteries. We define by $q_i^{(k)}$ the load (or utilization factor) of DPs of class k of cell i in the reduced network. We know that the reduced network is stable if and only if for all i

$$\sum_{k=1}^K q_i^{(k)} = \sum_{k=1}^K \frac{\lambda_i^{(k)} + \sum_{j=1}^N \mu_j^{(k)} q_j^{(k)} P_{j,i}^{(k)}}{\mu_i^{(k)}} < 1.$$

We now focus on the batteries in isolation. For this case, we consider that the battery of cell i, which has arrival rate from outside α_i and leakage rate β_i, receives a traffic $\sum_{k=1}^{K} \lambda_i^{(k)}$ from the server of the same cell. We denote by p_i the load of the battery in isolation of cell i. We know that the battery in isolation of cell i is stable if and only if

$$p_i = \frac{\alpha_i}{\beta_i + \sum_{k=1}^{K} \lambda_i^{(k)}} < 1.$$

We now define the hyperstability of a EPN.

Definition 1. *A EPN is hyperstable if its reduced network and all the batteries in isolation are stable.*

In the following result, we show that a EPN that is hyperstable, is stable. The proof is available in Appendix B.

Proposition 2. *If a EPN is hyperstable, then it is stable.*

In the previous result, we give sufficient conditions for the stability of a EPN. In the next section, we consider feed forward networks and we will see that the stability of the EPN can be fully characterized.

4 Feed Forward Networks

In this section, we focus on feed forward networks. In this kind of networks, there are no cycles, i.e., a job that has been served by a given cell does not return to that cell. Note that for a general topology, the algorithms presented in [5] and [7] cannot be extended to networks with EP and multiple classes of DP.

For these networks, we can invoke the results of Theorem 1 and Proposition 1 to conclude that (i) there exists a solution of the fixed point problem defined in Eq. 1 and Eq. 2 and (ii) if the fixed poing is stable, the distribution of jobs in this network has a product form. In this section, we show that the loads of the servers and of the batteries can be characterized for feed forward networks.

We notice that, for a feed forward EPN, the load of class k jobs in the server of cell i is given by

$$\rho_i^{(k)} = \frac{\lambda_i^{(k)} + \sum_{j \in PRE(i)} \mu_j^{(k)} \rho_j^{(k)} P_{j,i}^{(k)} (\gamma_j)^{c_j^{(k)}}}{\mu_n^{(k)}}, \tag{14}$$

where $PRE(i)$ is the set of predecessor cells of i. As a result, one can compute the value of $\rho_i^{(k)}$ using only the values of the load of the servers and batteries of the predecessor cells.

The load of the battery of cell i in an feed forward network is given by

$$\gamma_i = \frac{\alpha_i}{\beta_i + \sum_{k=1}^{K} \mu_i^{(k)} \rho_i^{(k)} (1 - d_i^{(k)}) \sum_{m=0}^{c_i^{(k)}-1} (\gamma_i)^m}, \tag{15}$$

where the value of $\rho_i^{(k)}$ are obtained from Eq. 14. Hence, if the load of the batteries and of the servers of the predecessor cells is known, the computation of the load of the battery of cell i is reduced to calculate the roots of a polynomial of degree $\max\{c_i^{(1)}, \ldots, c_i^{(K)}\}$.

From the previous reasoning, instead of solving a fixed point problem, one can construct a manner to compute the load of the batteries and the servers in an feed forward starting from the initial cells, i.e., those that do not have any predecessor, and continuing using the topological order of the network.

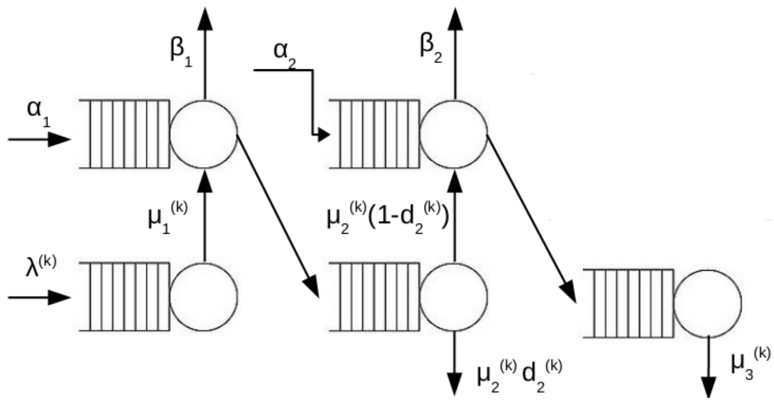

Figure 2: Example of a directed feed forward EPN with three cells.

4.1 Example

We now analyze an example of a feed forward EPN and we aim to show that the values of the loads of the servers and the batteries can be easily obtained. The network under consideration is depicted in Figure 2. It is a EPN with three cells. The server of cell 1 is the only one that receives traffic from outside the network. Upon service in the server of cell 1, jobs of class k that find $c_1^{(k)}$ packets in the battery are routed to cell 2. In cell 2, jobs of class k leave the system with probability $d_2^{(k)}$ and they do so with no energy requirement. Finally, upon service in cell 2, the DPs that find enough packets in the EP queue are routed to cell 3, where they leave the system with probability one when they have been served.

We consider that there are two classes and $c_i^{(1)} = 1$ and $c_i^{(2)} = 2$, for all cell i. The service rate is equal to one for all the servers and classes and $\lambda_1^{(1)} = 0.2$ and $\lambda_1^{(2)} = 0.3$. The probability of leaving the system in cell 2 is $d_2^{(1)} = 0.25$ and $d_2^{(2)} = 0.5$. The leakage rate of all the batteries is equal to 1 and the arrival rates to the batteries are $\alpha_1 = \alpha_2 = 1.2$ and $\alpha_3 = 0.8$.

Since this EPN is feed forward, we can obtain the values of the loads of the batteries and the servers so as to analyze its stability. Thus, we start with cell 1 and we obtain that

$$\rho_1^{(1)} = \frac{\lambda_1^{(1)}}{\mu_1^{(1)}} = \frac{0.2}{1} = 0.2$$

$$\rho_1^{(2)} = \frac{\lambda_1^{(2)}}{\mu_1^{(2)}} = \frac{0.3}{1} = 0.3,$$

and

$$\gamma_1 = \frac{\alpha_1}{\beta_1 + \rho_1^{(1)}\mu_1^{(1)} + \lambda_1^{(2)}\rho_1^{(2)}(1+\gamma_1)} = \frac{1.2}{1 + 0.2 + 0.3(1+\gamma)}.$$

The last expression is a quadratic function with the following roots: 0.7015 and -5.7015. Therefore, we have obtained the loads of cell 1 and we conclude that it is stable since $\rho_1^{(1)} + \rho_1^{(2)} = 0.5 < 1$ and $\gamma_1 = 0.7015 < 1$.

We now focus on cell 2. For this cell, we have that

$$\rho_2^{(1)} = \frac{\rho_1^{(1)} \cdot \mu_1^{(1)} \cdot (\gamma_1)^{c_1^{(1)}}}{\mu_2^{(1)}} = \frac{0.2 \cdot 1 \cdot 0.7015}{1} = 0.1403,$$

$$\rho_2^{(2)} = \frac{\rho_1^{(2)} \cdot \mu_1^{(2)} \cdot (\gamma_1)^{c_1^{(2)}}}{\mu_2^{(2)}} = \frac{0.3 \cdot 1 \cdot 0.7015^2}{1} = 0.1477.$$

and

$$\begin{aligned}\gamma_2 &= \frac{\alpha_2}{\beta_2 + \rho_2^{(1)}\mu_2^{(1)}(1-d_2^{(1)}) + \rho_2^{(2)}\mu_2^{(2)}(1-d_2^{(2)})(1+\gamma_2)} \\ &= \frac{1.2}{1 + 0.1403 \cdot 0.75 + 0.1477 \cdot 0.5(1+\gamma_2)},\end{aligned}$$

where the roots of the last equations are 0.6783 and -23.9723. According to the obtained values, the server and the battery of cell 2 are stable since $\rho_2^{(1)} + \rho_2^{(2)} = 0.1403 + 0.1477 = 0.288$ and $\gamma_2 = 0.6783$.

Regarding the battery of cell 3, since all the DP traffic leaves the system,

$$\gamma_3 = \frac{\alpha_3}{\beta_3} = \frac{0.8}{1} = 0.8,$$

and the load of the server is

$$\rho_3^{(1)} = \frac{\rho_2^{(1)}\mu_2^{(1)}(\gamma_2)^{c_2^{(1)}}}{\mu_3^{(1)}} = \frac{0.1403 \cdot 0.6783}{1} = 0.0951,$$

$$\rho_3^{(2)} = \frac{\rho_2^{(2)}\mu_2^{(2)}(\gamma_2)^{c_2^{(2)}}}{\mu_3^{(2)}} = \frac{0.1477 \cdot 0.6783^2}{1} = 0.0679.$$

Hence, the server and the battery of cell 3 are also stable since $\rho_3^{(1)} + \rho_3^{(2)} = 0.0951 + 0.0679 = 0.1631 < 1$ and $\gamma_3 = 0.8 < 1$. And, as a result, the EPN of Figure 2 is stable.

5 Conclusions

We have studied a multi class extension of the Energy Packet Network (EPN) model where classes of jobs are differentiated according to the number of EPs required for a successful sent to the following cell, the mean service times and the probabilities of moving through the network and leaving the system. Jobs that leave the system do not have any energy requirement and the servers operate under one of the following disciplines: FIFO, PS or LIFO-PR. We show that the steady-state distribution of jobs has a product form if the solution of a fixed point problem satisfies the stability conditions. We study the existence of a solution to the derived fixed point problem and we show that, assuming ergodicity, if the fixed point solution is stable, then it is unique. Furthermore, we explore the stability of the EPN and we provide sufficient conditions for the system to be stable. Finally, we focus on feed forward networks and we show that the loads of the queues can be fully characterized.

As a future work, we are interested in extending these results to servers with state dependent service rate μ_i. Such a model includes multiple servers queues and in particular Infinite Server ones. The proof we present here assumes that the total service rate is not state dependent. Another interesting question for further research is the existence of necessary and sufficient conditions for the stability of our model. Another possible research line that this work has opened is to study the convergence of the fixed point iteration algorithm of the problem defined in Eq. 1 and Eq. 2. This model can be extended to consider load-balancing techniques, as in [19, 23], where idle queues can poll another queues. Another interesting extension of this model consists of studying how fragmentation of jobs affects in the performance of the system as we can divide large DP packets with a large value of $c_i^{(k)}$ into smaller DP packets requiring less energy.

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A Proofs of Lemma 1, Lemma 2 and Lemma 3

Proof of Lemma 1: The proof consists in algebraic manipulations for the three types of servers.

LIFO-PR First consider an arbitrary LIFO server, we have: $M_j^{(l)}(x_j) = 1_{\{||x_j||>0\}}\mu_j^{(l)}1_{\{r_{j,1}=l\}}$. Thus,

$$M_j^{(l)}(x_j \oplus e^{(l)}) = \mu_j^{(l)}$$

Moreover, $\frac{g_j(x_j \oplus e^{(l)})}{g_j(x_j)} = \rho_j^{(l)}$ for a LIFO-PR queue. Therefore the relation holds.

PS We have:

$$M_j^{(l)}(x_j) = \mu_j^{(l)} \frac{x_{j,l}+1}{||x_j||+1} \quad and \quad \frac{g_j(x_j \oplus e^{(l)})}{g_j(x_j)} = \frac{(||x_j||+1)\rho_j^{(l)}}{x_{j,l}+1}$$

Once again the relation is true.

FIFO We have $M_j^{(l)}(x_j) = 1_{\{||x_j||>0\}}\mu_j 1_{\{r_{j,1}=l\}}$. Adding one customer at the head of the queue makes the two step functions equal to 1:

$$M_j^{(l)}(x_j \oplus e^{(l)}) = \mu_j,$$

while $\frac{g_j(x_j \oplus e^{(l)})}{g_j(x_j)}$ for a FIFO queue is equal to $\rho_i^{(l)}$. Noting that for a FIFO queue $\mu_i^{(l)} = \mu_i$ for all class index l, the relation also holds. □

Proof of Lemma 2: Again, the proof consists in algebraic manipulations for the three types of servers. Let

$$\sum_{l=1}^{K} M_j^{(l)}(x_j) - \sum_{l=1}^{K} \frac{g_j(x_j \ominus e^{(l)})}{g_j(x_j)} \lambda_j^{(l)} E(\vec{X} \ominus e_j^{(l)}) - \sum_{i=1}^{N}\sum_{l=1}^{K} \mu_i^{(l)} \rho_i^{(l)} \frac{g_j(x_j \ominus e^{(l)})}{g_j(x_j)} (\gamma_i)^{c_i^{(l)}} P_{i,j}^{(l)} E(\vec{X} \ominus e_j^{(l)}) = \Delta.$$

We want to prove that: $\Delta = 0$.

LIFO-PR First consider an arbitrary LIFO-PR server of cell j. We have that:

$$\sum_{l=1}^{K} M_j^{(l)}(x_j) = \sum_{l=1}^{K} 1_{\{||x_j||>0\}} \mu_j^{(l)} 1_{\{r_{j,1}=l\}},$$

while we have:

$$\sum_{l=1}^{K} \frac{g_j(x_j \ominus e^{(l)})}{g_j(x_j)} E(\vec{X} \ominus e_j^{(l)}) = \sum_{l=1}^{K} \frac{1}{\rho_j^{(l)}} 1_{\{||x_j||>0\}} 1_{\{r_{j,1}=l\}}.$$

Thus, after factorization:

$$\Delta = 1_{\{||x_j||>0\}} \left[\sum_{l=1}^{K} 1_{\{r_{j,1}=l\}} \left(\mu_j^{(l)} - \frac{\lambda_j^{(l)} + \sum_{i=1}^{N}\sum_{l=1}^{K} \mu_i^{(l)} \rho_i^{(l)} P_{i,j}^{(l)} (\gamma_i)^{c_i^{(l)}}}{\rho_j^{(l)}} \right) \right].$$

Therefore $\Delta = 0$ due to Eq. 1.

FIFO Consider now a FIFO queue. We have:

$$M_j^{(l)}(x_j) = 1_{\{||x_j||>0\}} \mu_j^{(l)} 1_{\{r_{j,1}=l\}} \ and \ \frac{g_j(x_j \ominus e^{(l)})}{g_j(x_j)} E(\vec{X} \ominus e_j^{(l)}) = \frac{1}{\rho_j^{(l)}} 1_{\{||x_j||>0\}} 1_{\{r_{j,\infty}=l\}}.$$

Note that the constraint for FIFO queues come from the step functions which are not equal. However, as all the service rates are equal we have:

$$\sum_{l=1}^{K} M_j^{(l)}(x_j) = 1_{\{||x_j||>0\}} \mu_j^{(l)} 1_{\{r_{j,1}=l\}} = 1_{\{||x_j||>0\}} \mu_j = 1_{\{||x_j||>0\}} \mu_j \sum_{l=1}^{K} 1_{\{r_{j,\infty}=l\}}.$$

Therefore we can factorize to obtain:

$$\Delta = 1_{\{||x_j||>0\}} \left[\sum_{l=1}^{K} 1_{\{r_{j,\infty}=l\}} \left(\mu_j^{(l)} - \frac{\lambda_j^{(l)} + \sum_{i=1}^{N} \sum_{l=1}^{K} \mu_i^{(l)} \rho_i^{(l)} P_{i,j}^{(l)} (\gamma_i)^{c_i^{(l)}}}{\rho_j^{(l)}} \right) \right].$$

Again $\Delta = 0$ due to Eq. 2

PS For an arbitrary PS queue (say j), he have: $M_j^{(l)}(x_j) = \mu_j^{(l)} \frac{x_{j,l}}{||x_j||} 1_{\{||x_j||>0\}}$ and

$$\frac{g_j(x_j \ominus e^{(l)})}{g_j(x_j)} E(\vec{X} \ominus e_j^{(l)}) = \frac{x_{j,l}}{||x_j||} \frac{1}{\rho_j^{(l)}} 1_{\{||x_j||>0\}}.$$

After factorization,

$$\Delta = 1_{\{||x_j||>0\}} \left[\sum_{l=1}^{K} \frac{x_{j,l}}{||x_j||} \left(\mu_j^{(l)} - \frac{\lambda_j^{(l)} + \sum_{i=1}^{N} \sum_{l=1}^{K} \mu_i^{(l)} \rho_i^{(l)} P_{i,j}^{(l)} (\gamma_i)^{c_i^{(l)}}}{\rho_j^{(l)}} \right) \right].$$

Because of Eq. 2 $\Delta = 0$ and the proof is complete. □

Proof of Lemma 3: Consider again the relation we want to prove:

$$\sum_{j=1}^{N} \sum_{l=1}^{K} \lambda_j^{(l)} + \sum_{j=1}^{N} \alpha_j = \sum_{j=1}^{N} \beta_j \gamma_j + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{K} \sum_{m=0}^{c_j^{(l)}-1} \mu_j^{(l)} \rho_j^{(l)} (\gamma_j)^m P_{j,i}^{(l)} + \sum_{j=1}^{N} \sum_{l=1}^{K} \mu_j^{(l)} \rho_j^{(l)} d_j^{(l)}.$$

Remember Eq. 2, move the denominator of the r.h.s. to the l.h.s and make a summation for all queue index i, we obtain:

$$\sum_{j=1}^{N} \alpha_j = \sum_{j=1}^{N} \beta_j \gamma_j + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{K} \gamma_j P_{j,i}^{(l)} \rho_j^{(l)} \mu_j^{(l)} \sum_{m=0}^{c_j^{(l)}-1} (\gamma_j)^m$$

Thus, combining both equations, we get:

$$\sum_{j=1}^{N} \sum_{l=1}^{K} \lambda_j^{(l)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{K} \sum_{m=0}^{c_j^{(l)}-1} \mu_j^{(l)} \rho_j^{(l)} (\gamma_j)^m P_{j,i}^{(l)} (1-\gamma_j) + \sum_{j=1}^{N} \sum_{l=1}^{K} \mu_j^{(l)} \rho_j^{(l)} d_j^{(l)}.$$

Now consider Eq. 1, move the denominator of the r.h.s. to the l.h.s and make a summation for all queue index i and class index k, we get:

$$\sum_{j=1}^{N} \sum_{l=1}^{K} \mu_j^{(l)} \rho_j^{(l)} = \sum_{j=1}^{N} \sum_{l=1}^{K} \lambda_j^{(l)} + \sum_{i=1}^{N} \sum_{l=1}^{K} \sum_{j=1}^{N} \mu_i^{(l)} \rho_i^{(l)} P_{i,j}^{(l)} (\gamma_i)^{c_i^{(l)}}$$

Taking into account the normalization of routing probability: for all j and l, $d_j^{(l)} + \sum_{i=1}^{N} P_{j,i}^{(l)} = 1$, we obtain after cancelation of terms, exchanging some indices and factorization:

$$\sum_{j=1}^{N} \sum_{l=1}^{K} \sum_{i=1}^{N} \mu_j^{(l)} \rho_j^{(l)} P_{j,i}^{(l)} (1 - (\gamma_j)^{c_j^{(l)}}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{K} \sum_{m=0}^{c_j^{(l)}-1} \mu_j^{(l)} \rho_j^{(l)} (\gamma_j)^m P_{j,i}^{(l)} (1-\gamma_j),$$

which is obviously true because

$$(1-\gamma_j)\sum_{m=0}^{c_j^{(l)}-1}(\gamma_j)^m = 1-(\gamma_j)^{c_j^{(l)}}.$$

$\square$

B Proof of Proposition 2

We first show that, if $p_i < 1$ for all i, then the batteries of the EPN are all stable, i.e., $\gamma_i < 1$ for all i. First, we note that

$$\sum_{m=0}^{c_i^{(k)}-1}\gamma_i = 1+\gamma_i+(\gamma_i)^2+\cdots+(\gamma_i)^{c_i^{(k)}-1} > 1 \tag{16}$$

since $\gamma_i > 0$. Besides, for all $k = 1, \ldots, K$ and $i = 1, \ldots, N$,

$$\mu_i^{(k)}\rho_i^{(k)} = \lambda_i^{(k)} + \sum_{k=1}^{K}\rho_i^{(k)}\mu_i^{(k)}(\gamma_i)^{c_i^{(k)}-1} > \lambda_i^{(k)}. \tag{17}$$

Using these properties, the desired result is proven in the following way:

$$\gamma_i = \frac{\alpha_i}{\beta_i + \sum_{k=1}^{K}\mu_i^{(k)}\rho_i^{(k)}\sum_{m=0}^{c_i^{(k)}-1}(\gamma_i)^m} \overset{Eq.\ 16}{<} \frac{\alpha_i}{\beta_i+\sum_{k=1}^{K}\mu_i^{(k)}\rho_i^{(k)}} \overset{Eq.\ 17}{<} \frac{\alpha_i}{\beta_i+\sum_{k=1}^{K}\lambda_i^{(k)}} = p_i < 1,$$

and the stability of the batteries of the EPN follows.

We now concentrate on the stability of the servers. First, we notice that the load of class k jobs of the reduced network is

$$\vec{q}^{\,(k)} * \vec{\mu}^{\,(k)}\left(I - P^{(k)}\right) = \vec{\lambda}^{(k)},$$

where $*$ denotes the component wise product of two vectors, $P^{(k)}$ is the transition probability of the network of class k jobs and $\vec{q}^{\,(k)} = (q_i^{(k)})_{i=1,\ldots,N}$, $\vec{\mu}^{(k)} = (\mu_i^{(k)})_{i=1,\ldots,N}$ and $\vec{\lambda}^{(k)} = (\lambda_i^{(k)})_{i=1,\ldots,N}$. Since the network is open, we know that $(I - P^{(k)})$ is non singular and non negative. Therefore,

$$\vec{q}^{\,(k)} * \vec{\mu}^{(k)} = \vec{\lambda}_k\ (I-P^{(k)})^{-1}. \tag{18}$$

For the EPN we consider, we have that, for all $k = 1, \ldots, K$, and $i = 1, \ldots, N$,

$$\rho_i^{(k)} = \frac{\lambda_i^{(k)} + \sum_{j=1}^{N}\mu_j^{(k)}\rho_j^{(k)}P_{j,i}^{(k)}(\gamma_j)^{c_j^{(k)}}}{\mu_i^{(k)}} < \frac{\lambda_i^{(k)}+\sum_{j=1}^{N}\mu_j^{(k)}\rho_j^{(k)}P_{j,i}^{(k)}}{\mu_i^{(k)}}$$

because the batteries are stable, i.e., $\gamma_j < 1$ for all j. Due to the non negativity of the matrix $(I - P^{(k)})^{-1}$, this is equivalent to say that, for all k,

$$\vec{\rho}^{\,(k)} * \vec{\mu}^{(k)} < \vec{\lambda}^{(k)}\ (I-P^{(k)})^{-1},$$

which, according to Eq. 18, gives that $\rho_i^{(k)} < q_i^{(k)}$ for all k and i and the stability of the servers of the EPN follows since the reduced network is stable, i.e., $\sum_{k=1}^{K} q_i^{(k)} < 1$ for all i.