Efficiency of Symmetric Nash Equilibria in Epidemic Models with Confinements*

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Abstract. We consider a non-cooperative game in the SIR model with confinements. Each member of the population is a player whose strategy is her probability of being protected from the epidemic. We assume that for each player, there is a cost of infection per unit time and a cost of being confined, which is linear and decreasing on her confinement strategy. The total cost is defined as the sum of her confinement and infection costs. We present a method for computing a symmetric Nash equilibrium for this game and study its efficiency. We conclude that the Nash equilibrium we obtain leads to fewer confinements than the strategy that minimizes the cost of the entire population.

Keywords: SIR model · Symmetric Nash equilibrium · Efficiency.

1 Introduction

The recent epidemic caused by COVID19 disease shows the great importance of mathematical models applied in this field. Indeed, these models can be used to analyze how a population will behave in an epidemic as well as the effect of several factors on the evolution of the spread of the disease.

Non-cooperative game-theory studies the behavior of self-interested agents (or players) that are in interaction [11]. A crucial concept in this field is the Nash equilibrium which is defined as the set of strategies such that none of the players gets benefit from a unilateral deviation. The Nash equilibrium appears in a variate of applications such as complex stochastic networks [1]. The SIR model is a stochastic network in which each member of the population under consideration belongs to one of the following states: susceptible (S), infected (I) or recovered (R). It was introduced in [9] and it has been considered in a large number of works since then (see the monographs [5, 2]).

We consider the SIR model with confinements. This means that the susceptible population can be protected from getting the disease. We consider that the size of the population is N. We formulate a non-cooperative game in which each member of the population is a player. The strategy that each player selects is her

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confinement strategy, which consists of the probability of being protected from the epidemic. When this probability is equal to zero, we say that the player is fully exposed to the epidemic, whereas when it is equal to one, it is completely confined. Associated to each infected member of the population, there is a cost per unit time. We consider that there is also a cost of confinement associated to the susceptible population, which is linear and decreasing with the confinement strategy. The cost of each player is thus the sum of her infection and confinement costs. We consider a problem with an infinite time horizon and discounted cost.

We are interested in calculating the symmetric Nash equilibrium of this game. First, we formulate a Markov Decision Process to calculate the best response strategy of a player to the set of strategies of the rest of the players. Using value iteration and a simple fixed-point algorithm, we present how to compute a pure Nash equilibrium.

We also formulate a global optimization problem whose goal is to find the optimal confinement strategy, i.e., the confinement strategy to be applied to the entire susceptible population in order to minimize the total cost of the population. We also formulate this problem as a Markov Decision Process. Finally, we compare the optimal confinement strategy with the Nash equilibrium strategy. Our first conclusion is that both strategies are very similar. However, in the Nash equilibrium strategy achieves complete confinement (i.e. the situation where a player is completely protected from the epidemic) more often than the global minimum strategy. Moreover, the Nash equilibrium switches from full exposure to full protection in a line which does not depend on the proportion of susceptible population. This situation is not achieved in the global minimum strategy, where this change is given in a switching curve.

Several game-theory based models have been studied considering vaccinations in the SIR models, for instance [8, 6, 10, 3]. However, confinements have been only studied using mean-field games in [4, 13] and considering that the entire population can control the contact rate. Our work is different as we consider game with N players and only susceptible population can control her interaction with the others.

2 Model Description

We analyze the SIR model in which a population of N people evolve over time. We consider that time is discrete. In the SIR model, each of the people belonging to the population under study is in one of the following three states: susceptible (S), infected (I) or recovered (R). We denote by $m_S(t)$, $m_I(t)$ and $m_R(t)$ the proportion of the population that is in each state.

We now describe the dynamics of this population. An individual encounters other individual in a time slot with probability γ. If an individual that is susceptible encounters an infected individual, then it becomes infected. Moreover, an infected individual becomes recovered in the next time slot (i.e., it gets recovered in a time slot) with probability ρ. We consider that R is an absorbing state, which means that the recovered population does not change her state.

We consider that the susceptible population can avoid to get the infection by choosing her confining strategy π. More precisely, a strategy π is a function that represents the exposure probability of the susceptible population to the infection, i.e. it is the probability that a susceptible individual is exposed to the infection at time slot t. For instance, if $\pi(t) = 0$, the susceptible population gets confined at time t, which means that they cannot get the infection at time slot t; whereas if $\pi(t) = 1$, the susceptible population is fully exposed to the infection. Thus, we have that $\pi : \mathbb{N} \to [0, 1]$.

In Figure 1, we represent the Markov chain describing the dynamics of an individual in this model.

$$\text{S} \xrightarrow{\quad \gamma m_I(t)\pi(t) \quad} \text{I} \xrightarrow{\quad \rho \quad} \text{R}$$

Fig. 1. The dynamics of an individual in the epidemic model under consideration. Each individual has three possible states: S (susceptible), I (infected) and R (recovered).

We focus on the evolution over time of the proportion of people in each state, which is described by the following equation:

$$\begin{cases} m_S(t+1) = m_S(t) - \gamma m_S(t) m_I(t) \pi(t) \\ m_I(t+1) = m_I(t) + \gamma m_S(t) m_I(t) \pi(t) - \rho m_I(t) \\ m_R(t+1) = m_R(t) + \rho m_I(t). \end{cases} \tag{1}$$

From this expression we derive several properties. For instance, we see that the proportion of people in state R is non-decreasing and also that, when $\pi(t) = 0$, the proportion of people in state S is constant and the proportion of people in state I is decreasing. Throughout this paper, we assume that $(m_S(0), m_I(0), m_R(0))$ is fixed. We also note that

$$(m_S(t), m_I(t), m_R(t)) = \left\{ \left(\frac{i}{N}, \frac{j}{N}, \frac{N-i-j}{N} \right) : i + j \leq N \right\}.$$

In the next section, we present a non-cooperative game for this model and in the following one, we analyze the efficiency of the Nash equilibrium.

3 Formulation of the Non-Cooperative Game

3.1 Game Description

We consider a non-cooperative game in the SIR model with confinements that we presented above. In this game, each individual of the population is a player that can choose her confinement strategy, that is, each player can select, in each time slot, her probability of being protected from the infection (or confinement probability). We denote by $\pi_i(t)$ the confinement probability of Player i and by $\pi_{-i}(t)$ the confinement probability of the entire population except for Player i, with $i = 1, \ldots, N$. We denote by π_i the vector of the confinement probability chosen by Player i in each time slot and by π_{-i} the vector of the confinement probability chosen by the rest of the players.

We consider that an infected player incurs a cost of $c_I > 0$ per unit of time. Moreover, we assume that there is also a confinement cost for each player that depends linearly on her strategy. More precisely, when at time slot t the Player i selects the confinement strategy $\pi_i(t)$, there is a confinement cost which equals $c_L - \pi_i(t)$, where $c_L \geq 1$. As a result, the cost of Player i at time slot t is the sum of her confinement cost and her infection cost. We denote by $x_s^{\pi_i,\pi_{-i}}(t)$ the probability that Player i is in state s at time slot t, where $s \in \{S, I, R\}$. Therefore, if we denote by $C_i(\pi_i, \pi_{-i})$ the total cost of Player i is given by

$$C_i(\pi_i, \pi_{-i}) = \sum_{t=0}^{\infty} \delta^t((c_L - \pi_i(t))x_S^{\pi_i,\pi_{-i}}(t) + c_I x_I^{\pi_i,\pi_{-i}}(t)), \quad \text{(COST-GAME)}$$

where $\delta \in (0, 1)$.

Remark 1. We would like to remark that [12] analyzed this model but considering the finite horizon case and they formulate a mean-field game (i.e., they consider that the number of players tends towards infinity). Our model differs significantly because we are considering a discounted cost infinite horizon case and, moreover, the number of players is finite and equal to N.

The best response of the Player i to π_{-i} is the confinement strategy that minimizes the above expression. That is,

$$BR_i(\pi_{-i}) = \arg\min_{\pi_i} C_i(\pi_i, \pi_{-i}) \quad \text{(BR-i)}$$

A symmetric Nash equilibrium is a strategy such that none of the players gets benefit from unilateral deviation. This means that π is a symmetric Nash equilibrium if, for all $i = 1, \ldots, N$

$$\pi = BR_i(\pi), \quad \text{(NASH-EQ)}$$

or alternatively, if for all $i = 1, \ldots, N$ and any other confinement strategy $\tilde{\pi}$,

$$C_i(\pi, \pi) \leq C_i(\tilde{\pi}, \pi).$$

The existence of a Nash equilibrium of this game follows from [7]. The computation of a Nash equilibrium is carried out using the set of instructions presented in Algorithm 1. Indeed, when the algorithm converges, we conclude that (NASH-EQ) is satisfied by π because all the players are symmetric (and therefore $BR_i(\pi) = BR_j(\pi)$ for $i \neq j$ and for every π). Therefore, in that case we can conclude that a pure Nash equilibrium has been found.

Algorithm 1 Fixed-point algorithm to compute a Nash equilibrium

Require: π_{-i}
  repeat
    $\tilde{\pi} \leftarrow \pi_{-i}$
    $\pi \leftarrow BR_i(\pi_{-i})$
  until $\pi = \tilde{\pi}$

3.2 Markov Decision Process Formulation

To obtain the best response of Player i to π_{-i}, we formulate a Markov Decision Process. To simplify the presentation, we consider that the size of the rest of the population is N (i.e., we consider a population of size $N+1$). The state of the Markov Decision Process is given by $(x, \frac{i}{N}, \frac{j}{N})$, where $x \in \{S, I, R\}$ and $i + j \neq N$, i.e., the first component represents the state of Player i and the rest of the components the possible values of the proportion of the susceptible and infected population. For a fixed strategy π, the strategy that minimizes (BR-i) (i.e., which is the best response of Player i to π) satisfies the following Bellman equations:

$$
\begin{aligned}
V^*_{k+1}\left(S, \frac{i}{N}, 0\right) =& \min_{\pi_i} [c_L - \pi_i], \quad i = 0, 1, \ldots, N, \\
V^*_{k+1}\left(S, 0, \frac{i}{N}\right) =& \min_{\pi_i} \Bigg[c_L - \pi_i + \delta \left(\gamma \frac{j}{N} \pi_i V^*_k \left(I, 0, \frac{i}{N}\right)\right. \\
&\left.\left. + \rho \frac{j}{N} V^*_k \left(S, 0, \frac{j-1}{N}\right)\right)\right], i = 1, \ldots, N, \\
V^*_{k+1}\left(S, \frac{i}{N}, \frac{j}{N}\right) =& \min_{\pi_i} \Bigg[c_L - \pi_i + \delta \left(\gamma \frac{j}{N} \pi_i V^*_k \left(I, \frac{i}{N}, \frac{j}{N}\right)\right. \\
& + \gamma \frac{j}{N} \pi \left(\frac{i}{N}, \frac{j}{N}\right) V^*_k \left(I, \frac{i-1}{N}, \frac{j+1}{N}\right) \\
&\left.\left. + \rho \frac{j}{N} V^*_k \left(S, \frac{i}{N}, \frac{j-1}{N}\right)\right)\right], i, j = 1, \ldots, N, \\
V^*_{k+1}\left(I, \frac{i}{N}, 0\right) =& c_I, \quad i = 0, 1, \ldots, N, \\
V^*_{k+1}\left(I, 0, \frac{i}{N}\right) =& c_I + \delta \rho \frac{i}{N} V^*_k \left(I, 0, \frac{i-1}{N}\right), \quad i = 1, \ldots, N \\
V^*_{k+1}\left(I, \frac{i}{N}, \frac{j}{N}\right) =& c_I + \delta \left(\gamma \frac{j}{N} \pi \left(\frac{i}{N}, \frac{j}{N}\right) V^*_k \left(I, \frac{i-1}{N}, \frac{j+1}{N}\right)\right. \\
&\left. + \rho \frac{j}{N} V^*_k \left(I, \frac{i}{N}, \frac{j-1}{N}\right)\right), \qquad i, j = 1, \ldots, N.
\end{aligned}
$$

From the first expression, we conclude that when none of the people of the rest of the population are infected (i.e., when $m_I = 0$), the best response of Player i

to π is $\arg\min_{\pi_i}[c_i - \pi_i]$, which gives one, i.e., Player i is never confined when there are no infected people. Moreover, from the second expression, we conclude that when none of the people of the rest of the population is susceptible, the best response is the value of π_i that minimizes

$$c_L - \pi_i + \delta\left(\gamma\frac{j}{N}\pi_i V_k^*\left(I, 0, \frac{i}{N}\right) + \rho\frac{j}{N}V_k^*\left(S, 0, \frac{j-1}{N}\right)\right).$$

Finally, from the third expression, we conclude that, when there are susceptible and infected people in the rest of the population, the best response of Player i to π is the value of π_i that minimizes

$$c_L - \pi_i + \delta(\gamma\frac{j}{N}\pi_i V_k^*\left(I, \frac{i}{N}, \frac{j}{N}\right) + \gamma\frac{j}{N}\pi(\frac{i}{N}, \frac{j}{N})V_k^*\left(I, \frac{i-1}{N}, \frac{j+1}{N}\right) + \rho\frac{j}{N}V_k^*\left(S, \frac{i}{N}, \frac{j-1}{N}\right)).$$

It is also remarkable that, in all the cases, the expression to be minimized so as to obtain the best response of Player i to π is linear in π_i. This implies that the best response, which can be in the interval $[0,1]$, will be one of the two following values: zero or one.

4 Efficiency of Nash equilibria

4.1 Global Optimum Confinement Strategy

We now focus on the global optimum confinement strategy, which is the value of π such that the cost of the population is minimized, i.e.,

$$\arg\min_{\pi}\sum_{t=0}^{\infty}\delta^t\left(c_I m_I(t) + (c_L - \pi)m_S(t)\right). \qquad \text{(GLOBAL-OPT)}$$

We know that the global optimum confinement strategy satisfies the following Bellman equations:

$$\begin{aligned}
V_{k+1}^*(0,0) &= 0\\
V_{k+1}^*\left(\frac{i}{N}, 0\right) &= \min_{\pi}[\frac{i}{N}(c_L - \pi)], \quad i = 1, \dots, N\\
V_{k+1}^*\left(0, \frac{i}{N}\right) &= c_I\frac{i}{N} + \delta\left(\rho V_k^*\left(0, \frac{i-1}{N}\right)\right), \quad i = 1, \dots, N\\
V_{k+1}^*(\frac{i}{N}, \frac{j}{N}) &= \min_{\pi}[c_I\frac{j}{N} + (c_L - \pi)\frac{i}{N} + \delta(\rho V_k^*\left(\frac{i}{N}, \frac{j-1}{N}\right)\\
&\qquad + \gamma\frac{j}{N}\pi V_k^*\left(\frac{i-1}{N}, \frac{j+1}{N}\right))], \qquad i, j = 1, \dots, N.
\end{aligned}$$

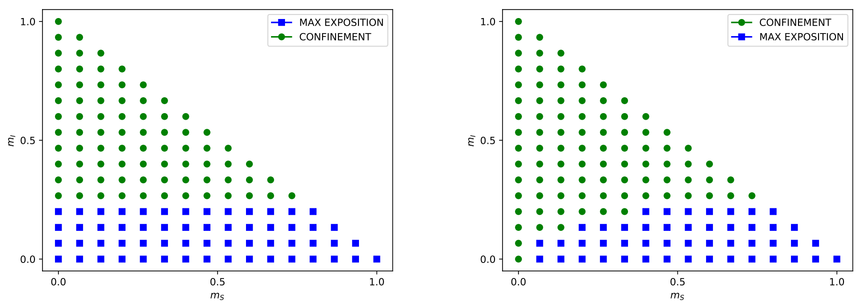

Fig. 2. Nash equilibrium strategy (left) and global optimum strategy (right) with N=15 and $c_I = 6$. The green dots represent that the probability of being protected from the epidemic is one, whereas blue squares that this probability is zero.

4.2 Efficiency Analysis

We now study the efficiency of the Nash equilibria of the formulated game, i.e., we compare the confinement strategy of (NASH-EQ) with that of (GLOBAL-OPT). We have performed extensive numerical experiments and here we present an example to illustrate the observed properties of all our experiments[1]. We consider $N = 15$, $c_I = 6$, $c_L = 2$, $\delta = 0.99$, $\gamma = 0.6$ and $\rho = 0.4$. The left plot of Figure 2 shows the confinement strategy of the Nash equilibrium we obtained with Algorithm 1. We observe that the Nash equilibrium consists of being completely exposed to the epidemic (which is represented as MAX EXPOSITION in the plot) when the proportion of infected population is smaller or equal to 0.25 and being completely protected from the epidemic (which is represented as CONFINEMENT in the plot) otherwise. However, in the right plot of Figure 2, we observe that the switching curve in which the optimal policy changes from being completely exposed to the epidemic to being completely confined is not a straight line (as in the Nash equilibrium case). Another interesting property of this experiment is that although both strategies are very similar, the Nash equilibrium strategy leads to less confinement (i.e., green dots are fewer in the left plot than in the right plot).

5 Future Work

This work is the first step of the research we plan to carry out in the future. Indeed, we would like to provide an analytical efficiency study of this model. Furthermore, we plan to study the conditions under which Algorithm 1 converges to a Nash equilibrium. Finally, we are interested in extending this model by considering that a recovered individual can become susceptible.

[1] The code to reproduce the experiments of this section is available at https://github.com/josudoncel/StudentsCode/tree/main/MaiderSanchezJimenez

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