Product form solution for the steady-state distribution of a Markov chain associated with a general matching model with self-loops

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Abstract. We extend the general matching graph model to deal with matching graph where every node has a self loop. Thus the states on the Markov chain are associated with the independent sets of the matching graph. We prove that under i.i.d. arrivals assumptions the steady-state distribution of the Markov chain has a product form solution.

1 Introduction

Intuitively, a Matching model describes the waiting times suffered by items before they match and disappear immediately once they are matched. It is an easy representation of multiple types Rendez-Vous between items. Following [1] a Matching model is a triple (G, Φ, μ) formed by

1. a matching graph $G = (\mathcal{V}, \mathcal{E})$ which is an undirected graph whose vertices in $\mathcal{V}$ are classes of items and whose edges in $\mathcal{E}$ models the allowed matching of items. G is called the compatibility graph or the Matching graph.
2. Φ is a matching policy. It states the couple of items which is chosen upon arrival when an arriving item matches one or more types of item already waiting.
3. a distribution of probability μ to model the arrivals of items. Alternatively one can consider a collection of Poisson processes for a continuous-time model.

The Matching graph represents the classes of items and the compatibility among classes of items. Upon arrival, an item is queued if there are not compatible items present in the system. A matching occurs when two (or more) compatible items are present and it is performed according to the matching discipline. Typical matching disciplines are First Come First Match (an analog of First Come First Served in this approach) or Match the Longest Queue. Once they are matched, both items leave the system immediately (no need for service). Note that even if a Matching model has a queueing theory flavor, the items play the roles of both customers and servers. In some sense it has also some links with two well known stochastic models: networks with positive and negative customers and stochastic Petri nets. In a network with positive and negative customer proposed by Gelenbe in [2] a negative customer can provoke the instantaneous deletion of a positive customer but it is never queued in the network. In [3] the deletion depends on the type as in a matching. In a stochastic Petri net, tokens wait until they match but the places where the tokens match are usually associated with non negative delays [4]. Under some structural conditions, stochastic Petri nets may have a product form steady-state solution (see for instance [5, 6]). Even if it possible to associate a Petri net with a matching graph model, our results differ from [5, 6].

Despite its simple formulation, Matching models were not so simple to analyze. Assuming independent Poisson arrivals of items, and FCFM discipline the model is associated with an infinite Markov chain. Under these assumptions, a necessary condition of stability and a product form solution were proved in [7] and [1]. Moreover we recently established that there exists some performance paradox for FCFM matching models [8]. When one add new edges in the compatibility graph, one may expect that the expectation of the total number of customers decreases. In [8] we have given some examples which show that it is not always the case and we prove a sufficient condition for such a performance paradox to exist. Thus adding flexibility on the matching does not always result in a performance improvement.

The general matching model proposed in [7] and [1] was considering a general undirected matching graph G and it is assumed that the arrivals of items occur one at a time. It is important to avoid the confusion with Bipartite Matching Model (see for instance [9] and references therein) where the matching graph is bipartite and two items of distinct classes arrive at the same time. Bipartite Matching Models were motivated by analysis of the public housing [10]. In this model, households which apply for public housing and housings which become available both arrive over time. Once the matching is done, the housing is occupied for a long time period. Thus it is more convenient to represent them as an arrival streams of items rather than traditional servers in a queue. Another application studied in the literature was the kidney exchanges [11, 12]. The kidney exchange arises when a healthy person who wishes to donate a kidney is not compatible (blood types or tissue types) with the receiver. Two incompatible pairs (or maybe more) can form a cyclic exchange, so that each patient can receive a kidney from a compatible donor (see [13] and reference therein for a presentation of the problem and the modeling and algorithmic issues).

Here, we further extend the type of graph to represent the general matching. We assume that all the nodes in the matching graph have a self loop (see Fig. 1 for an example of such a graph) and this was clearly forbidden by the previous assumptions in [1]. Note that the results obtained in [7] and [1] are not valid anymore because of some technical details in the proofs of product form. It is required in these papers that the matching graph does not contain self loops.

Without self loops the Markov chain is infinite and the system stability has to be studied taking into account the matching discipline.

For our new model, the stability problem is not an issue. As all the nodes in the compatibility graph have a self loop, the system can contain at most one item of each type. Therefore the Markov chain associated with the population is finite. If the chain is irreducible, it is therefore ergodic (see more details in section 2). Such a model may represent how we can organize fair competition between players with roughly the same ranking (for instance ELO points for chess). Such an application was not possible for the model described in [1].

A more general result for matching with multigraph has recently been presented in [14]. In that paper a multigraph is defined as a graph where the self loops are allowed. Therefore the proof they proposed is more general than our result because the self loops are not mandatory leading to potential infinite Markov chains. However our proof is based on simpler arguments (balance equations rather than reversibility of an extended process followed by an aggregation) and we hope it has its own value.

The technical part of the paper is as follows. We begin in the next section with the notations. Section 3 is devoted to a simple example built for a matching graph with 4 nodes. We obtained the steady-state distribution and checked that a multiplicative solution holds. In Section 4, we prove that this product form solution holds for every matching graph with self loops for all the nodes.

2 Notation and Assumptions

Let $G = (\mathcal{V}, \mathcal{F})$ be the matching graph. Nodes in $\mathcal{V}$ are also denoted as letters. An ordered list of letters is called a word. Assume that x is a letter from $\mathcal{V}$, $\Gamma(x)$ is the set of neighbors of x in G. As all nodes in $\mathcal{V}$ carry a loop, we have $x \in \Gamma(x)$ for all node x.

Let m be an arbitrary word.

- $(m|x)$ is the word obtained by appending letter x at the end of word m while $(m+x)$ represents the set of words obtained by adding an x anywhere inside word m (even before word m).
- $|m|$ will denote the size of word m (i.e. the number of letters).
- $m(\psi)$ will be the letter of m at position ψ.
- $\Gamma(m) = \cup_{x \in m} \Gamma(x)$.
- $Pre(m, \psi)$ is the prefix of m with size ψ. The prefix of size 0 is the empty word denoted as E or $\emptyset$.
- Similarly $Suf(m, \psi)$ is the suffix of m with size ψ.
- Finally, $In(m, x, \psi)$ is the word of size $|m|+1$ built from m after the insertion of x at position ψ.

We consider a discrete time model. We assume i.i.d. arrivals of a letter at every time slot. We consider the First Come First Match policy or FCFM (sometimes denoted as the First Come First Served discipline in the literature). An arriving letter will be added at the end of the word if it does not match any letter in the current word. If the arriving letter matches one or several letters of the current word, both the oldest matching letter and the arriving letter vanish immediately.

α_i will denote the probability of arrival of letter i while α_0 is the probability that there is no arrival. The state of the system at time t is a word. Under these assumptions, the process $\{m_t\}$, with $t \in \mathbb{N}$ is a Discrete Time Markov chain. If $\alpha_0 > 0$, this Markov chain is aperiodic. Furthermore it is clear that all the states of the Markov chain are independent sets of the matching graph. Indeed, if two letters are neighbors in the matching graph, they cannot be in a state of the Markov chain. Furthermore, as all the nodes of the matching graph have a self loop, it is not possible to have several occurrences of the same letter in a node of the chain. This last property does not hold for matching models without loops (see for instance [7]). Therefore the Markov chain is finite (again this is not true in the model studied in [7] and [1]). Clearly if $\alpha_i > 0$ for all i, the chain is irreducible. Therefore we do not have to study the stability problem as in [1]. The chain is always ergodic.

For all subset S of nodes of G, α_S will denote the probability of arrival of letters in S:

$$\alpha_S = \sum_{x \in S} \alpha_x$$

To simplify the formulation of the steady-state distribution, we make the following remark:

Remark 1 *Let m be an arbitrary state and x be an arbitrary letter, such that $(m|x)$ is a valid state of the chain, we have $\Gamma(m+x) = \Gamma(m|x) = \Gamma(x|m)$. Remember that $\Gamma(x)$ is the set of neighbors of state x.*

3 Path of length 4

We begin with a simple example. We consider as a matching graph, a path of length 4 (usually denoted as $P4$) with loops on every node (see Fig. 1).

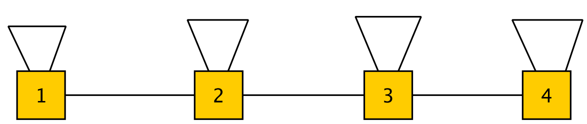

Fig. 1. Matching graph: $P4$.

We first build the Markov chain associated with this matching graph. The states are based on the independent sets of this graph. The states take into account the order of arrivals (remember that we consider FCFM discipline).

These states (and sets) contain up to two letters. A state is labelled by the letters which are included while E will denote the empty state. $(x|y)$ represents the state containing letter x followed by letter y. The associated independent set is $\{x, y\}$. Therefore set $\{x, y\}$ is associated with states $(x|y)$ and $(y|x)$.

This Markov chain has 11 states: E, 1, 2, 3, 4, $(1|4)$, $(4|1)$, $(1|3)$, $(3|1)$, $(2|4)$, $(4|2)$. The graph of the Markov chain obtained by the XBorne [15] tool is depicted in Fig. 2. We do not add the transition probabilities to make the figure more understandable. For the same reason, we do not draw either the loop on every state associated with the null arrival event (with probability α_0).

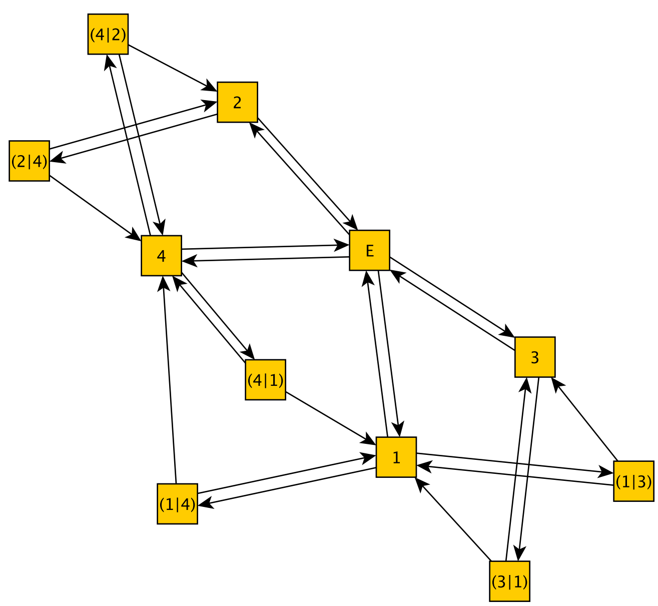

Fig. 2. Graph of the Markov chain associated with Matching graph $P4$.

We will write the global balance equations for an arbitrary node of the Markov chain, taking into account the graph properties of the nodes. We consider the following partition of the states of the chain based on their properties.

1. State E.
2. States associated with maximal independent sets. For this matching graph, a maximal independent set contains two letters and they are denoted $(x|y)$.
3. States which are neither the empty state nor a state associated with a maximal independent set. Here, these states are words with a single letter.

A similar partition will be used in the next section to make the proof in the general case.

To write the transition probability matrix, we consider that the states are in the following order E, 1, 2, 3, 4, (1|4), (4|1), (1|3), (3|1), (2|4), (4|2). The partition is added into the matrix presentation to make the block structure more visible.

$$P = \left[\begin{array}{c|cccc|cc|cc|cc}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\alpha_1+\alpha_2 & \alpha_0 & 0 & 0 & 0 & \alpha_4 & 0 & \alpha_3 & 0 & 0 & 0 \\
\alpha_1+\alpha_2+\alpha_3 & 0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_4 & 0 \\
\alpha_2+\alpha_3+\alpha_4 & 0 & 0 & \alpha_0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 \\
\alpha_3+\alpha_4 & 0 & 0 & 0 & \alpha_0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_2 \\
\hline
0 & \alpha_3+\alpha_4 & 0 & 0 & \alpha_1+\alpha_2 & \alpha_0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_3+\alpha_4 & 0 & 0 & \alpha_1+\alpha_2 & 0 & \alpha_0 & 0 & 0 & 0 & 0 \\
\hline
0 & \alpha_3+\alpha_4 & 0 & \alpha_1+\alpha_2 & 0 & 0 & 0 & \alpha_0 & 0 & 0 & 0 \\
0 & \alpha_2+\alpha_3+\alpha_4 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 \\
\hline
0 & 0 & \alpha_4 & 0 & \alpha_1+\alpha_2+\alpha_3 & 0 & 0 & 0 & 0 & \alpha_0 & 0 \\
0 & 0 & \alpha_3+\alpha_4 & 0 & \alpha_1+\alpha_2 & 0 & 0 & 0 & 0 & 0 & \alpha_0
\end{array}\right]$$

Consider now the global balance equations. Let us begin with a maximal independent set. Writing a balance equation for a state $(x|y)$ we get:

$$\pi(x|y) = \pi(x|y)\alpha_0 + \pi(x)\alpha_y,$$

from which we easily obtain for state $(x|y)$:

$$\pi(x|y) = \pi(x)\frac{\alpha_y}{1-\alpha_0}.$$

Now consider a state with one letter (x) . For instance consider State (3).

$$\pi(3)(1-\alpha_0) = \pi(1|3)(\alpha_1+\alpha_2) + \pi(E)\alpha_3 + \pi(3|1)\alpha_1$$

Using the relations we already obtained, we substitute $\pi(1|3)$ and $\pi(3|1)$:

$$\pi(3)(1-\alpha_0) = \pi(1)(\alpha_1+\alpha_2)\frac{\alpha_3}{1-\alpha_0} + \pi(E)\alpha_3 + \pi(3)\alpha_1\frac{\alpha_1}{1-\alpha_0}$$

One can check with some simple algebraic manipulations of these equations, that the solution we propose in Eq. 1 is the solution of the balance equations.

$$\begin{array}{ll}
\pi(x|y) & = \pi(x)\alpha_y/(1-\alpha_0) \\
\pi(1) & = \pi(E)\alpha_1/(\alpha_1+\alpha_2) \\
\pi(2) & = \pi(E)\alpha_2/(\alpha_1+\alpha_2+\alpha_3) \\
\pi(3) & = \pi(E)\alpha_3/(\alpha_2+\alpha_3+\alpha_4) \\
\pi(4) & = \pi(E)\alpha_4/(\alpha_3+\alpha_4)
\end{array} \tag{1}$$

and $\pi(E)$ is obtained by normalization.

Note that this solution is the solution proved by Moyal et al. in [7] for a general matching model without loops on the matching graph:

$$\pi(w_1|..|w_k) = C\prod_{i=1}^{k}\frac{\alpha(w(i))}{\alpha(\Gamma(w(1),...,w(i)))} \tag{2}$$

Thus one may expect that the multiplicative solution for the steady-state still holds under our assumptions.

4 Steady-State distribution

We now prove that the Markov chain associated with any matching graph which has a loop on all nodes has a steady state distribution which has a multiplicative form. To prove the theorem, we use the following characterization of the steady-state distribution.

Property 1 *Consider a Markov chain with state space $\mathcal{E}$ and transition matrix P. Let π be a finite measure (i.e. $||\pi|| < \infty$). If matrix Q defined by for all i and j in $\mathcal{E}$*

$$\pi(i)P[i,j] = \pi(j)Q[j,i]$$

is a stochastic matrix, then the steady state distribution of the Markov chain associated with P is obtained through normalization of π (i.e. $\pi/||\pi||$).

Remark 2 *If $P = Q$, the chain is reversible.*

Assuming that the multiplicative solution already known for matching graphs without loop still holds, one can formally obtain matrix Q and check if this matrix is stochastic. This is the key idea for the proof.

Remark 3 *As for all x we have $P[x,x] = \alpha_0$, then, by construction, $Q[x,x] = \alpha_0$.*

4.1 *P*4 revisited

Consider again the example of the Markov chain associated with a P4 matching graph. We compute Q^t (instead of Q to simplify the presentation) by computing the product with the probabilities $\pi()$ given in Eq. 1. For instance $Q[i,i] = \alpha_0$ for all i, and:

$$Q[(1|4),(1)] = \frac{\pi((1))}{\pi((1|4))}P[(1),(1|4)] = \frac{1-\alpha_0}{\alpha_4}\alpha_4 = 1-\alpha_0$$

We denote $\beta_0 = 1 - \alpha_0$ to simplify the matrix formulation.

$$Q^t = \left[\begin{array}{c|cccc|cc|cc|cc}
\alpha_0 & \alpha_1+\alpha_2 & \alpha_1+\alpha_2+\alpha_3 & \alpha_2+\alpha_3+\alpha_4 & \alpha_3+\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\alpha_1 & \alpha_0 & 0 & 0 & 0 & \beta_0 & 0 & \beta_0 & 0 & 0 & 0 \\
\alpha_2 & 0 & \alpha_0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_0 & 0 \\
\alpha_3 & 0 & 0 & \alpha_0 & 0 & 0 & 0 & 0 & \beta_0 & 0 & 0 \\
\alpha_4 & 0 & 0 & 0 & \alpha_0 & 0 & \beta_0 & 0 & 0 & 0 & \beta_0 \\
\hline
0 & \frac{\alpha_3(\alpha_3+\alpha_4)}{\beta_0} & 0 & 0 & \frac{\alpha_1(\alpha_3+\alpha_4)}{\beta_0} & \alpha_0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\alpha_4(\alpha_1+\alpha_2)}{\beta_0} & 0 & 0 & \frac{\alpha_1(\alpha_1+\alpha_2)}{\beta_0} & 0 & \alpha_0 & 0 & 0 & 0 & 0 \\
\hline
0 & \frac{\alpha_3(\alpha_3+\alpha_4)}{\beta_0} & 0 & \frac{\alpha_1(\alpha_2+\alpha_3+\alpha_4)}{\beta_0} & 0 & 0 & 0 & \alpha_0 & 0 & 0 & 0 \\
0 & \frac{\alpha_3(\alpha_1+\alpha_2)}{\beta_0} & 0 & \frac{\alpha_1^2}{\beta_0} & 0 & 0 & 0 & 0 & \alpha_0 & 0 & 0 \\
\hline
0 & 0 & \frac{\alpha_4^2}{\beta_0} & 0 & \frac{\alpha_2(\alpha_3+\alpha_4)}{\beta_0} & 0 & 0 & 0 & 0 & \alpha_0 & 0 \\
0 & 0 & \frac{\alpha_4(\alpha_1+\alpha_2+\alpha_3)}{\beta_0} & 0 & \frac{\alpha_2(\alpha_1+\alpha_2)}{\beta_0} & 0 & 0 & 0 & 0 & 0 & \alpha_0
\end{array}\right]$$

Clearly Q^t is column stochastic, thus Q is a stochastic matrix. And the result (i.e. Eq. 1) holds.

4.2 Main result

Let us now proceed with our main result on the steady-state distribution of the Markov chain associated with the matching graph. We first state the result. Before proceeding with the proof, we give some technical lemmas for the 3 types of nodes, as mentioned during the analysis of the $P4$ example.

Theorem 1 *Let G be a graph with a loop on each vertex. Let n be the number of nodes of G. Let $\alpha_0, \alpha_1, ..., \alpha_n$ be a proper distribution of probability. The steady-state distribution of the Markov chain associated with matching graph G has a multiplicative form:*

$$\pi(m) = C \frac{\prod_{\psi=1}^{n} \alpha_{m(\psi)}}{\prod_{\psi=1}^{n} \alpha(\Gamma(Pre(m, \psi)))}.$$

where C is a normalization constant equal to $\pi(E)$.

Corollary 1 *Assume that state $(m|x)$ exists, then we have:*

$$\pi(m|x) = \pi(m) \frac{\alpha_x}{\alpha(\Gamma(Pre(m|x)))}.$$

The proof of the theorem is based on the analysis of matrix Q for the three types of node. First we need to study the graph properties of the Markov chain. We begin with the description of the edges.

Lemma 1. *A state m of the Markov chain a positive number of letters (i.e. $|m| > 0$) only has transitions to itself (because of null arrival event) and to states with size $|m| + 1$, or $|m| - 1$.*

Proof: It is a clear consequence of the description of the effect of an arrival.

Lemma 2. *A state m of the Markov chain with a positive number of letters (i.e. $|m| > 0$) has only one transition to a state with size $|m| - 1$ and the loop transition with probability α_0. The other transitions lead to states with size $|m|+1$ Furthermore, if m is not a maximal independent set, for all the letters x such that $(m|x)$ exists, we have $P(m, m|x) = \alpha_x$.*

Proof: First, let m be a state with a positive number of letter. We can write $m = (j|x)$, where j is a word (one can have $j = E$)). There exists a unique transition from j to $(j|x)$ with rate α_x due to the arrival of a letter x and the FCFM matching discipline. Indeed, the only possibility to increase the size of the state is the arrival of a letter which must be the last one due to the FCFM discipline.

Finally according to the previous lemma, all the remaining transitions leads to states with one more letter. And if $(m|x)$ exists, the only transition from m to $(m|x)$ is the arrival of letter x and it has probability α_x.

Lemma 3. *[Global balance equation for a maximal independent set] Assume that the size of the maximal independent sets is at least 2. A state is a maximal independent set if it is a word $(m|x)$ where m is also an independent set which does not contain an x. The only transition (except the loop) entering such a state comes from state m and has probability α_x. There exist at least two outgoing transitions: the loop with rate α_0 and all the transitions provoked by arrivals which delete one letter in $(m|x)$. Thus the global balance equation for such a state is:*

$$(1-\alpha_0)\pi(m|x) = \pi(m)\alpha_x$$

Proof: Clearly, it exits a transition with probability α_x going from m to $(m|x)$ if x does not match with a letter of m. According to Lemma 1, the states which precede $(m|x)$ have size $|m|$, $|m|+1$ (due to the loop) or $|m|+2$. Clearly, the only transition entering $(m|x)$ from a state with a smaller number of letters is the transition going from m due to the FCFM discipline.

Let us prove now by contradiction that there does not exist any transition from a state $(m'|x)$ to a state $(m|x)$ with $|m'| = |m| + 2$. Assume that it is possible, then state $(m'|x)$ has a size larger than the size of $(m|x)$ and as an arrival provokes a transition from $(m'|x)$ to $(m|x)$, all the letters of $(m|x)$ are also in $(m'|x)$. Thus $(m|x)$ cannot be a maximal independent set as it is contained in a larger independent set.

Finally, state $(m|x)$ has an output degree which is at least 3. Indeed there exists a transition from $(m|x)$ to m since the arrival of x deletes the last letter x in the word (this is the effect of the loop on x in the matching graph). Furthermore, the arrivals of all the letters in m delete a letter of m and to not delete x. Therefore they provoke the transition to $(m'|x)$ with $|m'| = |m| - 1$. Finally we add the loop and the output degree is at least 3. The global balance equation is now trivial.

Property 2 *Let m be a state of the chain and x an arbitrary letter which is in $V \setminus \Gamma(m)$. We define the following subset of states:*

$$\Gamma^{-x}(m) = \{p \;\big|\; |p| = 1 + |m|, \text{ and } \; p = m + x\}$$

Intuitively, the arrival of letter x in state p provokes a transition from p to m because letter x is deleted. Such a subset is only defined when x is in $V \setminus \Gamma(m)$. Indeed, if the word has size $|m|+1$, it means that letter x does not interact with word m. Furthermore the cardinal of $\Gamma^{-x}(m)$ is $|m|+1$. Indeed, one can add letter x anywhere inside word m.

Lemma 4. *Let j be a state (i.e. a word) of size n. For all letter x in $V \setminus \Gamma(j)$, we have:*

$$\sum_{l \in \Gamma^{-x}(j)} Q(j,l) = \alpha_x.$$

As the proof of this lemma is technical, it is postponed after the proof of the main theorem.

Proof of the Theorem Let us now proceed with the proof the main theorem. Remember we partition the states of the Markov chain into three subsets:

1. Empty state E
2. Maximal Independent states
3. Other states

We make the proof for the three types of states:

- Empty State E: for all letter x we have by construction:

$$Q(E,x) = P(x,E)\frac{\pi(x)}{\pi(E)}$$

  From the definition of π in Theorem 1, $\pi(x) = \frac{\pi(E)\alpha_x}{\alpha(\Gamma(x))}$ and $P(x,E) = \alpha(\Gamma(x))$. After simplification,

$$Q(E,x) = \alpha_x$$

  Thus $\sum_{x\in V} Q(E,x) = \sum_{x\in V} \alpha_x = 1 - \alpha_0$ and $Q(j,j) = \alpha_0$. Therefore the row sum is equal to 1 for state E.
- Maximal Independent set: Let p be a maximal independent set. According to Lemma 3, there exists now only one entry in the column associated with p in P. Thus there exists only one entry in the row associated with p in Q. Let m be this predecessor of p and let x the letter appended to m to obtain p (i.e. $p = (m|x)$). By construction:

$$Q(p,m) = P(m,p)\pi(m)/\pi(p).$$

  From Lemma 3, $P(m,p) = \alpha_x$, and $(1-\alpha_0)\pi(p) = \alpha_x\pi(m)$. Thus, $Q(p,m) = 1-\alpha_0$. Furthermore $Q(p,p) = \alpha_0$. And all other entries of matrix Q for row p are 0. Therefore the row sum is equal to 1 for a state which is a maximal Independent set.
- Other states: we know due to Lemma 2 that, if it is not a maximal independent set, a state j of size n has one predecessor with size $n-1$,several predecessors of size $n+1$ and itself (with probability α_0). We will consider these three sets of predecessors in a separate way.
  - Let m the predecessor of size $n-1$. Let x the letter which has been appended (i.e. $j = m|x$). From Lemma 2 we have: $P(m,j) = \alpha_x$. From Corollary 1, we have

$$\pi(j) = \pi(m|x) = \frac{\pi(m)\alpha_x}{\alpha(\Gamma(m|x))}$$

    Thus

$$Q(j,m) = \alpha(\Gamma(m|x)) = \alpha(\Gamma(j))$$

- We now consider the predecessors of j which have size $n+1$. Let $H(j)$ be this set. We will partition this set of states according to the letter x which provokes the transition. Thus,

$$H(j) = \cup_{x \in V \setminus \Gamma(j)} \Gamma^{-x}(j).$$

As subsets $\Gamma^{-x}(j)$ do not intersect, this is a true partition. Thus,

$$\sum_{l \in H(j)} Q(j,l) = \sum_{x \in V \setminus \Gamma(j)} \sum_{l \in \Gamma^{-x}(j)} Q(j,l)$$

Technical Lemma 4 states that: $\sum_{l \in \Gamma^{-x}(j)} Q(j,l) = \alpha_x$. Thus,

$$\sum_{l \in H(j)} Q(j,l) = \sum_{x \in V \setminus \Gamma(j)} \alpha_x.$$

Combining both results, we get:

$$\sum_{l} Q(j,l) = \alpha_0 + \alpha(\Gamma(j)) + \sum_{l \in H(j)} Q(j,l) = \alpha_0 + \alpha(\Gamma(j)) + \sum_{x \in V \setminus \Gamma(j)} \alpha_x = \sum_{x \in V} \alpha_x = 1$$

And the proof is complete.

4.3 Proof of the technical lemma

We want to prove that for an arbitrary word j of size n and for all letter x in $V \setminus \Gamma(j)$, we have:

$$\sum_{l \in \Gamma^{-x}(j)} Q(j,l) = \alpha_x.$$

Let us first explain the terms involved in the summation and the way we combine them. By definition we have:

$$Q(j,l) = P(l,j)\pi(l)/\pi(j).$$

and by assumptions, the solution is:

$$\pi(l) = \pi(E) \frac{\prod_{\psi=1}^{n+1} \alpha_{l(\psi)}}{\prod_{\psi=1}^{n+1} \alpha(\Gamma(Pre(l,\psi)))}.$$

As $l \in \Gamma^{-x}(j)$, one can write $l = j + x$ and

$$\prod_{\psi=1}^{n+1} \alpha_{l(\psi)} = \alpha_x \prod_{\psi=1}^{n} \alpha_{j(\psi)}$$

Let us now study the denominator. Consider an arbitrary word $(j+x)$. There exists an index ψ (which can be 0) such that this word is $(Pre(j,\psi)|x|Suf(j,n-$

ψ)). This formulation allows to obtain the transition probability and the steady-state distribution for all the values of ψ and obtain an induction on partial sums on ψ

Let us begin the induction with $\psi = 0$ associated with $Q(j, (x|j))$. Clearly,

$$P((x|j), j) = \alpha(\Gamma(x)).$$

Now the denominator of $\pi(x|j)$ is according to the assumptions equal to $\prod_{\psi=1}^{n+1} \alpha(\Gamma(Pre((x|j), \psi)))$. It is easy to remark that:

$$\prod_{\psi=1}^{n+1} \alpha(\Gamma(Pre((x|j), \psi))) = \prod_{\psi=0}^{n} \alpha(\Gamma(x|Pre(j, \psi)))$$

This remark is used to simplify the factorization.

$$\pi(x|j) = \pi(E)\alpha_x \frac{\prod_{\psi=1}^{n} \alpha_{j(\psi)}}{\prod_{\psi=0}^{n} \alpha(\Gamma((x|Pre(j, \psi))))}$$

and,

$$\pi(j) = \pi(E) \frac{\prod_{\psi=1}^{n} \alpha_{j(\psi)}}{\prod_{\psi=1}^{n} \alpha(\Gamma(Pre(j, \psi)))}$$

and finally after cancellation of terms:

$$Q(j, (x|j)) = \alpha_x \alpha(\Gamma(x)) \frac{\prod_{\psi=1}^{n} \alpha(\Gamma(Pre(j, \psi)))}{\prod_{\psi=0}^{n} \alpha(\Gamma((x|Pre(j, \psi))))}$$

We now have to compute the term associated with $\psi = 1$ (remember that we want to make an induction on partial sums). The state of the chain is $(Pre(j, 1)|x|Suf(j, n-1))$. The transition from this state to j is provoked by the arrival of letters which match with x but which are not matched with the first letter of this word due to the FCFM matching discipline. More formally:

$$P((Pre(j, 1)|x|Suf(j, n-1)), j) = \alpha(\Gamma(x) \setminus \Gamma(Pre(j, 1))),$$

and

$$\pi((Pre(j, 1)|x|Suf(j, n-1))) = \pi(E)\alpha_x \frac{\prod_{\psi=1}^{n} \alpha_{j(\psi)}}{\alpha(\Gamma(Pre(j, 1))) \prod_{\psi=1}^{n} \alpha(\Gamma((x|Pre(j, \psi))))},$$

and finally after substation and cancellation, we get:

$$Q(j, (Pre(j, 1)|x|\ Suf(j, n-1))) = \alpha_x \alpha(\Gamma(x) \setminus \Gamma(Pre(j, 1))) \frac{\prod_{\psi=1}^{n} \alpha(\Gamma(Pre(j, \psi)))}{\alpha(\Gamma(Pre(j, 1))) \prod_{\psi=1}^{n} \alpha(\Gamma((x|Pre(j, \psi))))}.$$

We now compute the sum of these first two elements. After factorization:

$$Q(j, (x|j)) + Q(j, (\ Pre(j, 1)|x|Suf(j, n-1))) = \alpha_x \frac{\prod_{\psi=1}^{n} \alpha(\Gamma(Pre(l, \psi)))}{\prod_{\psi=1}^{n} \alpha(\Gamma((x|Pre(j, \psi))))} \left[\frac{\alpha(\Gamma(x))}{\alpha(\Gamma((x|Pre(j, 0))))} + \frac{\alpha(\Gamma(x) \setminus \Gamma(Pre(j, 1)))}{\alpha(\Gamma(Pre(j, 1)))} \right].$$

As $(Pre(j,0))$ is the empty word, we have $\mathit{\Gamma}((x|Pre(j,0)))) = \mathit{\Gamma}(x)$, and the first part of the summation simplifies.

$$Q(j,(x|j)) + Q(j,(Pre(j,1)|x|\ Suf(j,n-1))) =$$
$$\alpha_x \frac{\prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}(Pre(j,\psi)))}{\prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}((x|Pre(j,\psi)))} \left[1 + \frac{\alpha(\mathit{\Gamma}(x)\setminus \mathit{\Gamma}(Pre(j,1)))}{\alpha(\mathit{\Gamma}(Pre(j,1)))}\right].$$

Thus:

$$Q(j,(x|j)) + Q(j,(Pre(j,1)|x|\ Suf(j,n-1))) =$$
$$\alpha_x \frac{\prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}(Pre(l,\psi)))}{\prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}((x|Pre(j,\psi)))} \frac{\alpha(\mathit{\Gamma}(Pre(j,1)))+\alpha(\mathit{\Gamma}(x)\setminus \mathit{\Gamma}(Pre(j,1)))}{\alpha(\mathit{\Gamma}(Pre(j,1)))}.$$

For all words a and b, we have:

$$\alpha(\mathit{\Gamma}(a) \setminus \mathit{\Gamma}(b)) + \alpha(\mathit{\Gamma}(b)) = \alpha(\mathit{\Gamma}(a|b)).$$

Using $a = x$ and $b = Pre(j,1)$, after substitution we get:

$$Q(j,(x|j)) + Q(j,(Pre(j,1)|x|\ Suf(j,n-1))) =$$
$$\alpha_x \frac{\prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}(Pre(j,\psi)))}{\prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}((x|Pre(j,\psi)))} \frac{\alpha(\mathit{\Gamma}((x|Pre(j,1))))}{\alpha(\mathit{\Gamma}(Pre(j,1)))}.$$

$\alpha(\mathit{\Gamma}((x|Pr(j,1))))$ is in the numerator and the denominator. We cancel this term and we get:

$$Q(j,(x|j)) + Q(j,(Pre(j,1)|x|\ Suf(j,n-1))) =$$
$$\alpha_x \frac{\prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}(Pre(j,\psi)))}{\alpha(\mathit{\Gamma}(Pre(j,1))) \prod_{\psi=2}^{n} \alpha(\mathit{\Gamma}((x|Pre(j,\psi)))}.$$

Let us now consider the induction on the number of terms we add. Assume that we consider them and accumulate them according to the index of x in the word. Assume that we have proved that for all ψ between 0 and $\nu - 1$, we have stated that the summation of the first ν terms is equal to:

$$\frac{\alpha_x \prod_{\psi=1}^{n} \alpha(\mathit{\Gamma}(Pre(j,\psi)))}{\prod_{\psi=1}^{\nu-1} \alpha(\mathit{\Gamma}(Pre(j,\psi))) \prod_{\psi=\nu}^{n} \alpha(\mathit{\Gamma}((x|Pre(j,\psi)))}.$$

We now consider term with index ν. We have to compute $Q(j,(Pre(j,\nu)|x|Suf(j,n-\nu)))$ and add it to the previous partial sum. As usual due to the FCFM matching discipline

$$P((Pre(j,\nu)|x|Suf(j,n-\nu)),j) = \alpha(\mathit{\Gamma}(x) \setminus \mathit{\Gamma}(Pre(j,\nu))),$$

By assumption on the multiplicative solution for the steady-state:

$$\pi((Pre(j,\nu)|x|\ Suf(j,n-\nu))) =$$
$$\pi(E) \frac{\alpha_x \prod_{\psi=1}^{n} \alpha_{j(\psi)}}{\prod_{\psi=1}^{\nu} \alpha(\mathit{\Gamma}(Pre(j,\psi))) \prod_{\psi=\nu}^{n} \alpha(\mathit{\Gamma}((x|Pre(j,\psi))))}.$$

Thus,

$$Q(j,(Pre(j,\nu)|x|\ Suf(j,n-\nu))) =$$

$$\alpha(\Gamma(x)\setminus\Gamma(Pr(j,\nu)))\frac{\alpha_x\prod_{\psi=1}^{n}\alpha(\Gamma(Pre(j,\psi)))}{\prod_{\psi=1}^{\nu}\alpha(\Gamma(Pre(j,\psi)))\prod_{\psi=\nu}^{n}\alpha(\Gamma((x|Pre(j,\psi))))}.$$

After summation with the previous partial sum (given by the induction assumption) and factorization, we get that the new partial summation is equal to:

$$\frac{\alpha_x\prod_{\psi=1}^{n}\alpha(\Gamma(P(j,\psi)))}{\prod_{\psi=1}^{\nu-1}\alpha(\Gamma(Pre(j,\psi)))\prod_{\psi=\nu}^{n}\alpha(\Gamma((x|Pre(j,\psi)))}\left[1+\frac{\alpha(\Gamma(x)\setminus\Gamma(Pre(j,\nu)))}{\alpha(\Gamma(Pre(j,\nu)))}\right].$$

We use a similar argument to simplify:

$$1+\frac{\alpha(\Gamma(x)-\Gamma(Pre(j,\nu)))}{\alpha(\Gamma(Pre(j,\nu)))}=\frac{\alpha(\Gamma((x|Pre(j,\nu))))}{\alpha(\Gamma(Pre(j,\nu)))}.$$

After substitution the sum is equal to:

$$\frac{\alpha_x\prod_{\psi=1}^{n}\alpha(\Gamma(Pre(j,\psi)))}{\prod_{\psi=1}^{\nu-1}\alpha(\Gamma(Pre(j,\psi)))\prod_{\psi=\nu}^{n}\alpha(\Gamma((x|Pre(j,\psi))))}\ \frac{\alpha(\Gamma((x|Pre(j,\nu))))}{\alpha(\Gamma(Pre(j,\nu)))}=$$

$$\frac{\alpha_x\prod_{\psi=1}^{n}\alpha(\Gamma(Pre(j,\psi)))}{\prod_{\psi=1}^{\nu}\alpha(\Gamma(Pre(j,\psi)))\prod_{\psi=\nu+1}^{n}\alpha(\Gamma((x|Pre(j,\psi))))}.$$

Thus the induction holds. Now let us compute the sum of all the elements for ψ between 0 and n. According to the induction assumptions which is now proved, this sum is equal to

$$\alpha_x\frac{\prod_{\psi=1}^{n}\alpha(\Gamma(Pre(j,\psi)))}{\prod_{\psi=1}^{n}\alpha(\Gamma(Pre(j,\psi)))}=\alpha_x,$$

and the proof of the Lemma is complete.

5 Conclusions and Remarks

This paper is a sequel of [8] where we proved that adding a new edge in a matching graph without loops may lead to a performance paradox: the expectation of the total number of customers increase after the addition of the edge. See also [16] for an extended version of this paper.

Our aim was to prove or disprove the existence of the same paradox for matching graph with loops. The first step was to prove that the steady state solution has a multiplicative form. However even with this result, the existence of a paradox similar to the one shown in [8] is still an open problem as all the examples studied so far do not exhibit the same paradox we found in [8]. Note however that the chains we obtain with this new model are all finite while the chains studied in [16] are infinite.

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