On the Inefficiency of Atomic Routing Games over Parallel Links

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joint work with O. Brun

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November 10, 2023

Outline

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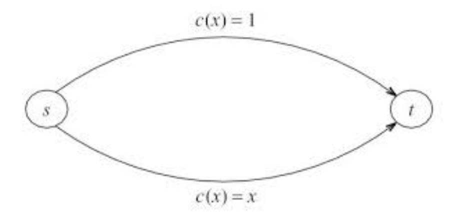
Pigou's example



Efficiency Analysis of Nash Equilibrium

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NE is inefficient in routing games because the PoA is large

[Roughgarden 2002, Haviv et al. 2007, Altmann et al 2011, Ayesta et al. 2011, Bell et al. 1983, Anselmi et al. 2010, Chen et al. 2009, Czumaj et al. 2022, Suri et al. 2004, Katsupias et al. 1999, Ghosh et al. 2021, ...]

Some recent works

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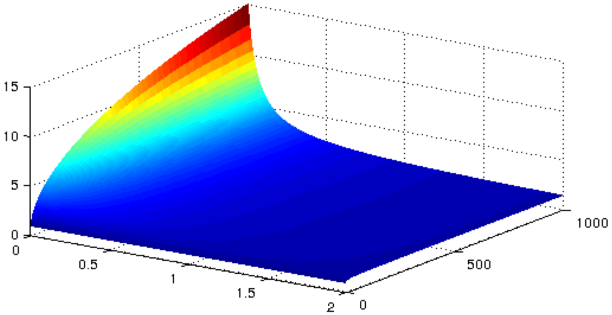
In practice the NE is efficient:

- [Monnot et al. 2017] Commuting times analysis of Singapore
- [Colini-Baldeschi 2020] Low and high traffic analysis

Conclusion: The PoA is a very pessimistic measure of the efficiency of non-cooperative routing games

Example

The ratio $\frac{\text{cost at NE}}{\text{optimal cost}}$ in system with two parameters



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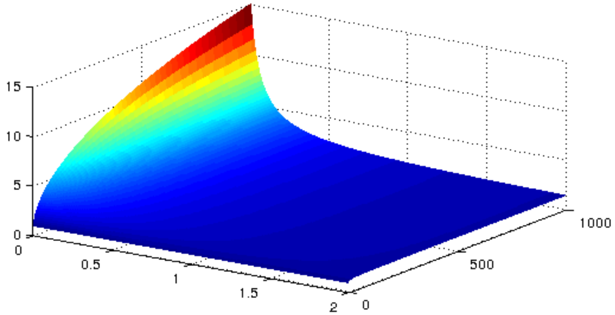

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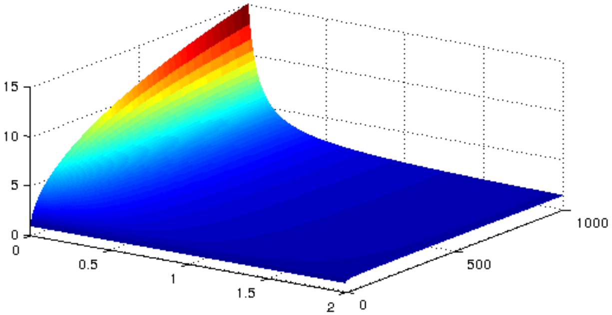

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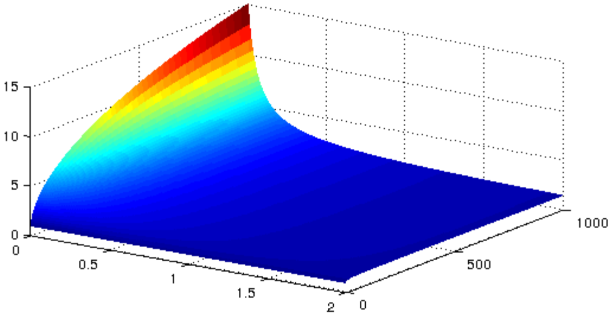

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Inefficiency ⇒ System of Parallel Links

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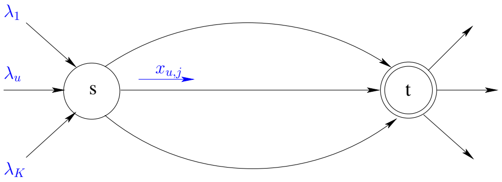

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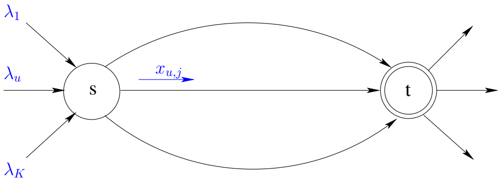

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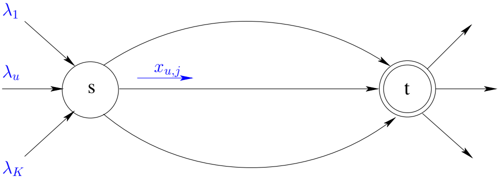

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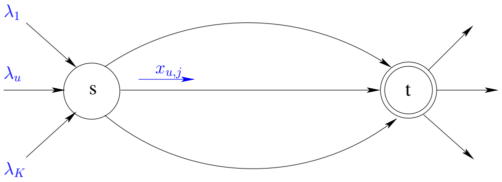

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Orda et al. 1993

Existence and uniqueness of NE for a very large family of $\phi()$

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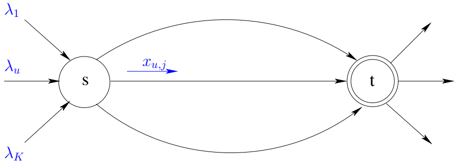

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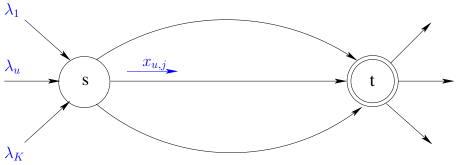

$$PoA = \sup_{\mathbf{p}} I_K(\mathbf{p})$$

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We are searching the maximum of a single parameter of the cost at NE strategy over the cost at the optimum strategy

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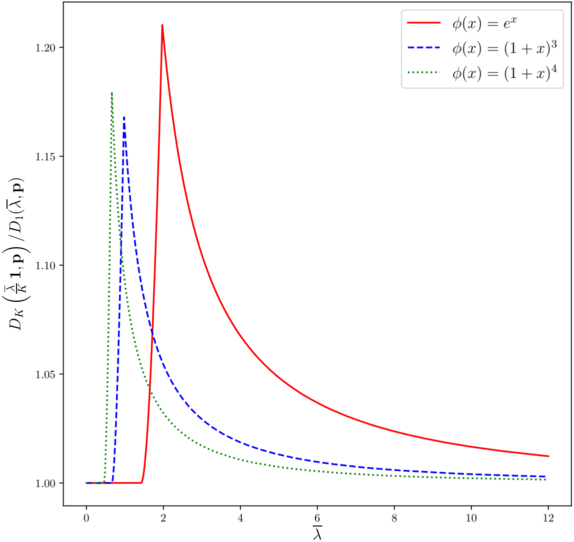

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- $D_K/D_1 = 1$ when both configurations use only cheap links
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We characterize the value of $\overline{\lambda}$ for which the NE strategy starts using all the links

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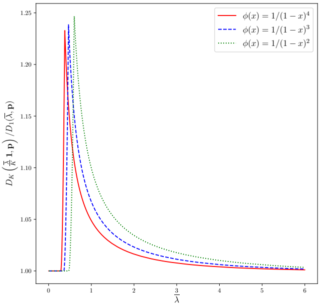

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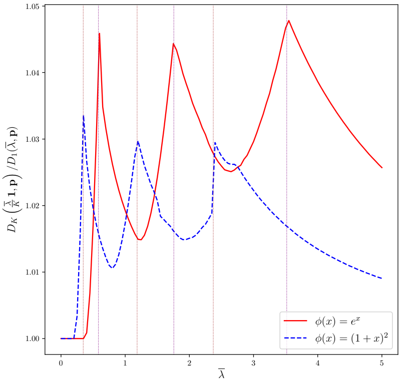
For more than two types of links, we observe numerically that there are peaks when the NE strategy starts using a new type of links.
$\Rightarrow$ There are several local maxima!



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Thanks for you attention