Performance Paradox of Dynamic Bipartite Matching Models

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Joint work with I. Iriondo (UPV/EHU)

NETGCOOP 2024. Lille, France.
October 9, 2024

Outline

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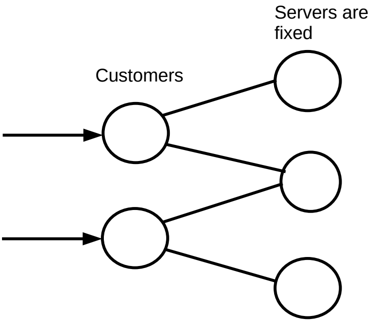
Introduction

Multiskilled systems (call centers)



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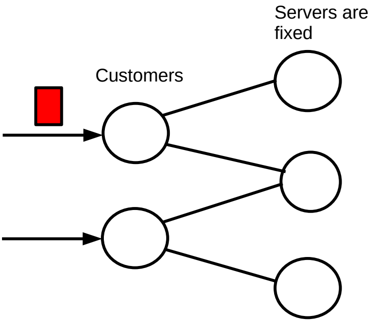

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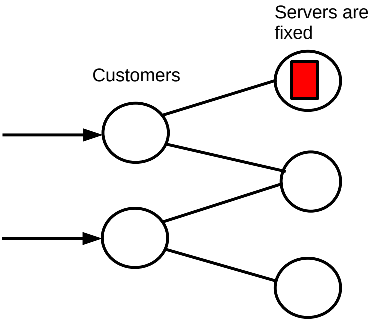

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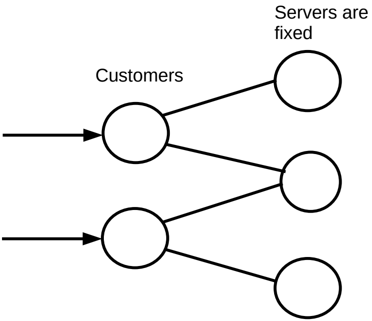

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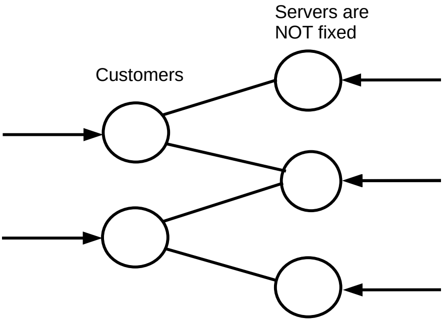

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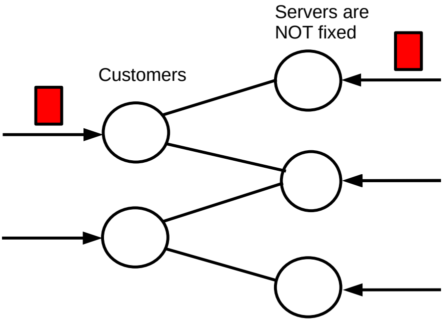

Matched customers and servers leave immediately

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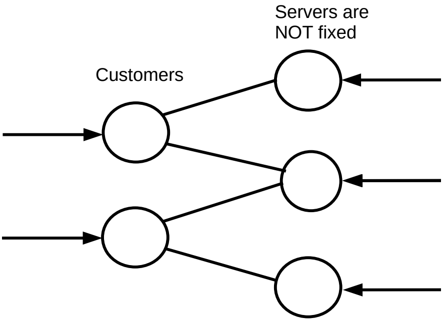

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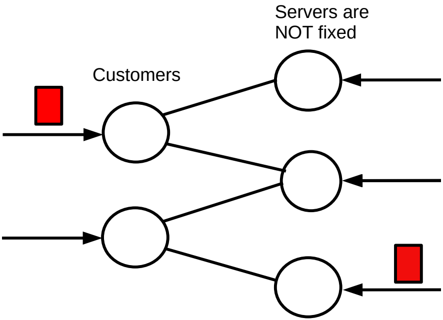

Unmatched customers and servers are stored in queues

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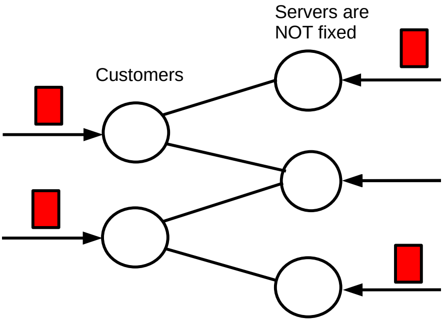

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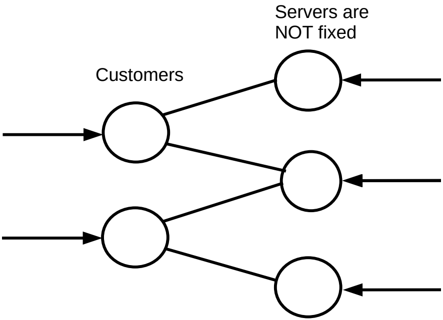

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Bipartite Matching Models

Defined by:

- Compatibility graph $\mathcal{G}$.

- Distribution of arrivals of customers and servers

- Matching policy ψ

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  Example: $\mathcal{C} = \{c_1, c_2\}$, $\mathcal{S} = \{s_1, s_2, s_3\}$ and
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  Example: $\vec{\alpha} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$

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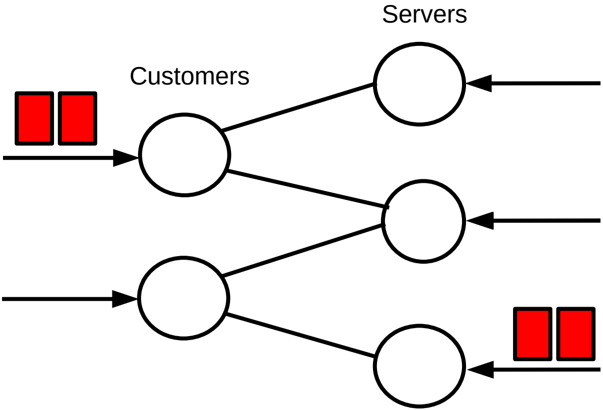
Matching Policy: Example



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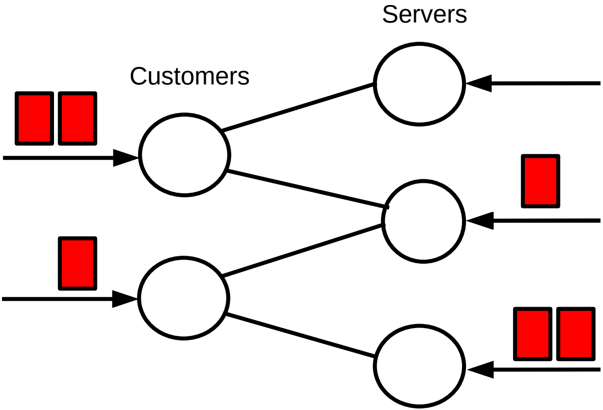

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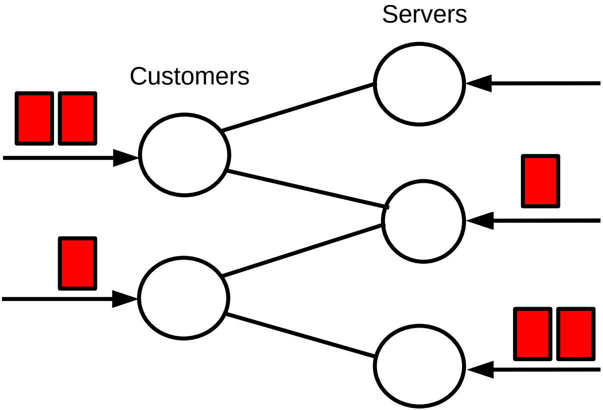

First-Come-First-Matched (FCFM)

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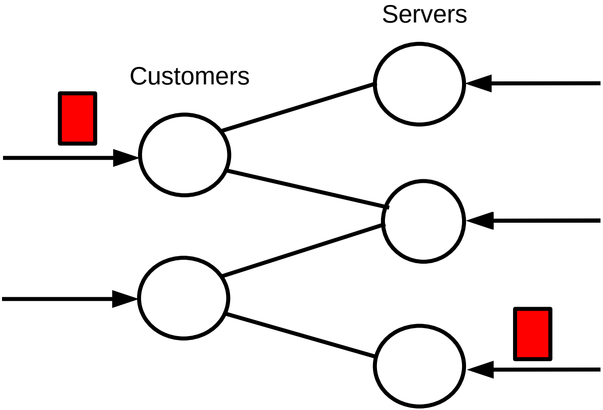

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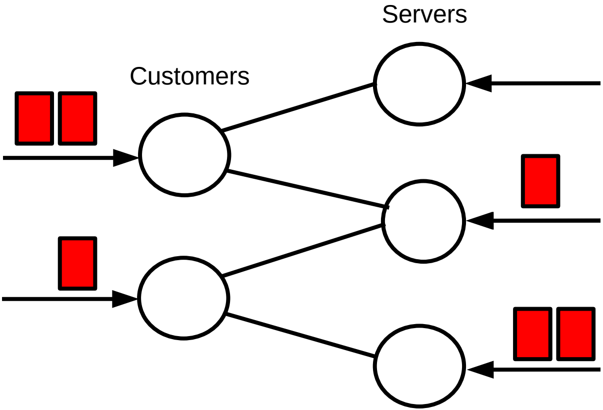

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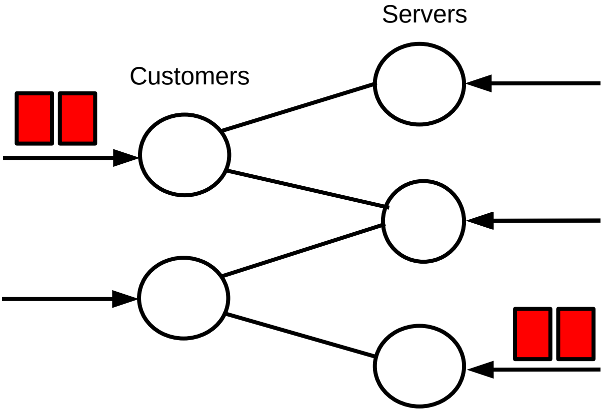

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- Distribution of arrivals of customers and servers (independent)
  $\vec{\alpha}$ for customers and $\vec{\beta}$ for servers

  Example: $\vec{\alpha} = (\alpha_1, \alpha_2)$ and $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$
  c_1 and s_3 arrive with probability $\alpha_1\beta_3$.

- Matching policy ψ

  How compatible customers and servers are matched?
  - In order of arrivals
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When $\mathcal{G}$, $(\vec{\alpha}, \vec{\beta})$ and ψ are fixed

The number of unmatched items is a Discrete Time Markov Chain.

Stability of Markov Chain

Busic, Gupta and Mairesse, 2013

For FCFM, the Markov chain is stable iff $\forall C \subset \mathcal{C}\ \forall S \subset \mathcal{S}$

$$\sum_{c_i \in C} \alpha_i < \sum_{s_i \in S(C)} \beta_i \text{ and } \sum_{s_i \in S} \beta_i < \sum_{c_i \in C(S)} \alpha_i,$$

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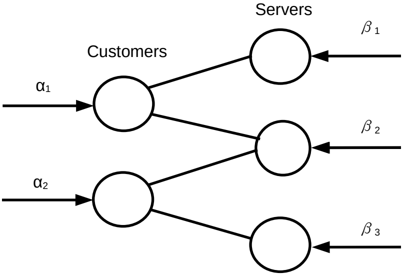

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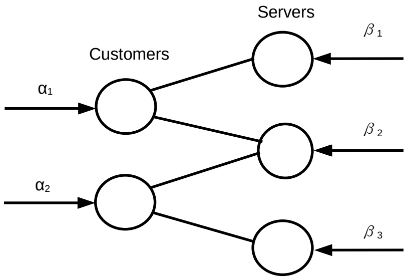

- $\alpha_1 < \beta_1 + \beta_2$
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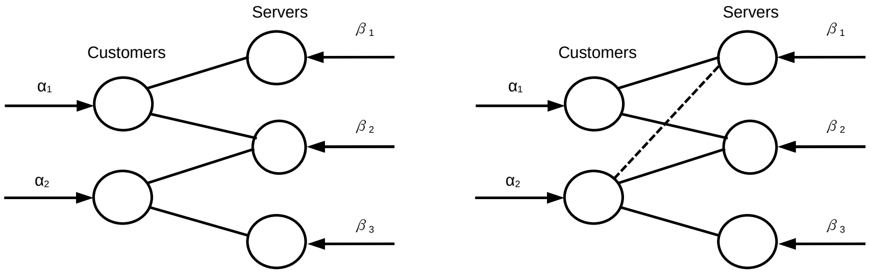
FCFM matching policy

Objective: study the influence of adding an edge to the compatibility graph



Model Description

FCFM matching policy

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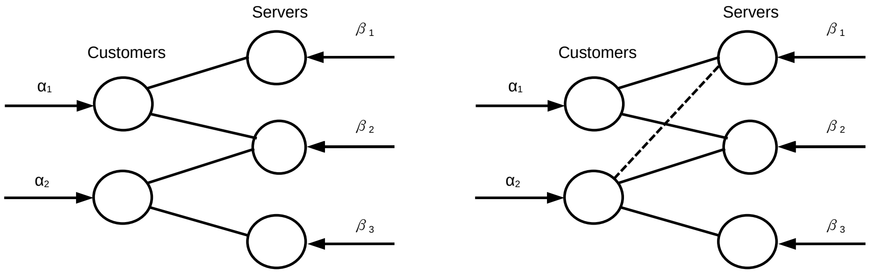

Definition: Performance paradox

Adding an edge the mean number of unmatched customers and servers increases (analogue of Braess paradox).

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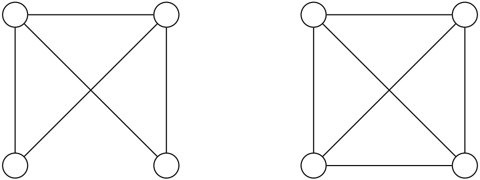

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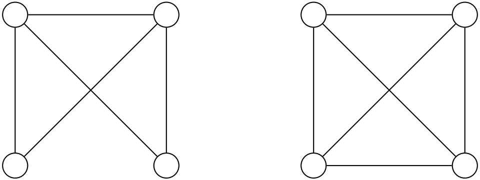

Necessary and sufficient conditions on the arrivals such that performance paradox exists in a quasicomplete graph under FCFM (Cadas et al, 2021) and under greedy policies (Busic et al, 2024)

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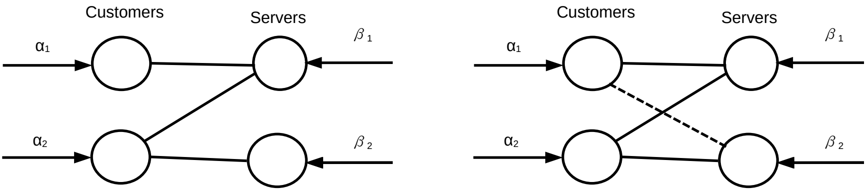

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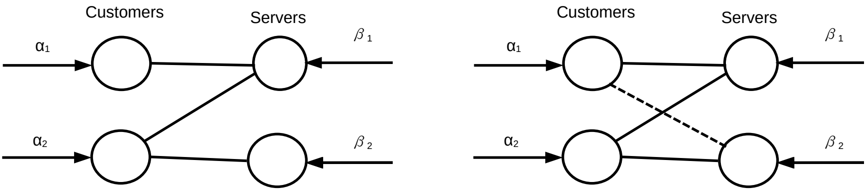
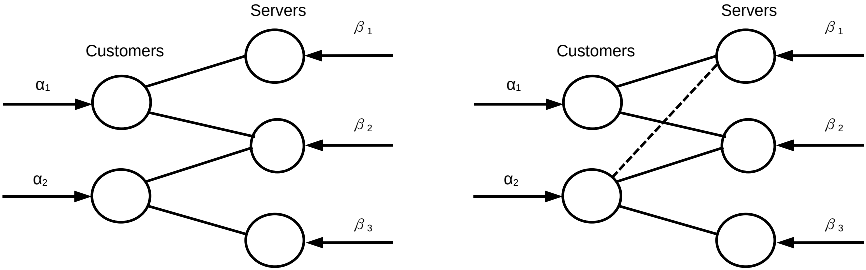


The performance paradox does NOT exist for this case

Performance Paradox Analysis

Second approach:



Performance Paradox Analysis

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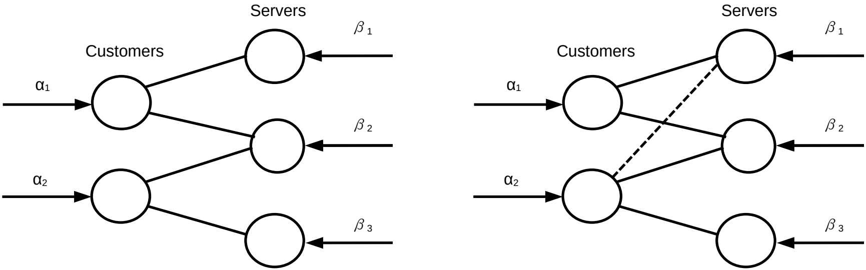
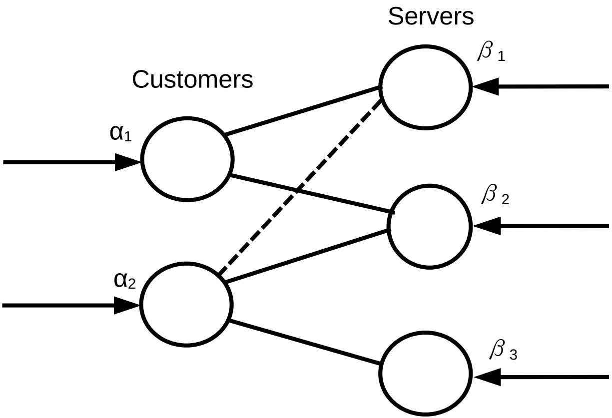


Under FCFM, the Markov chains of the unmatched elements are not difficult to analyze

Performance Paradox Analysis



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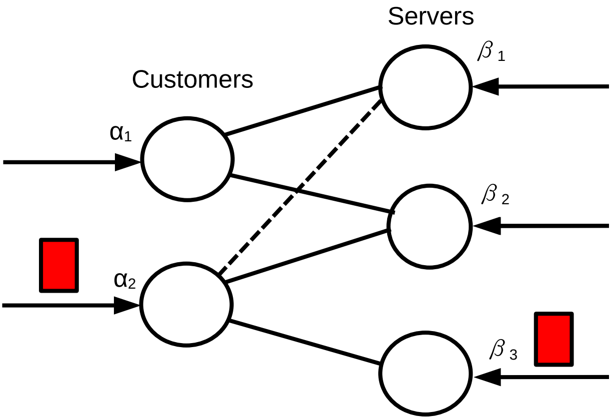

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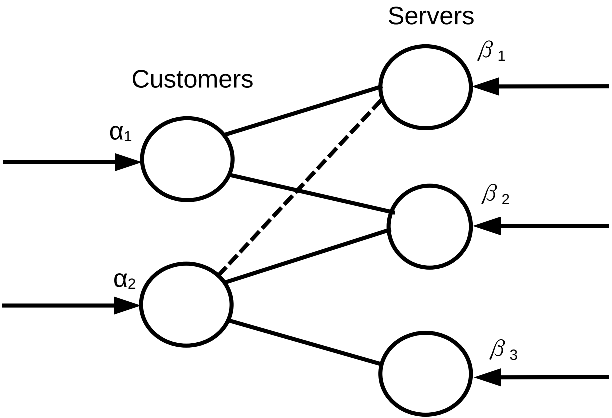

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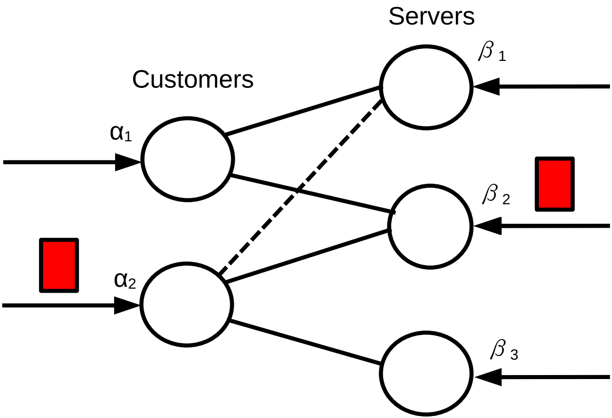

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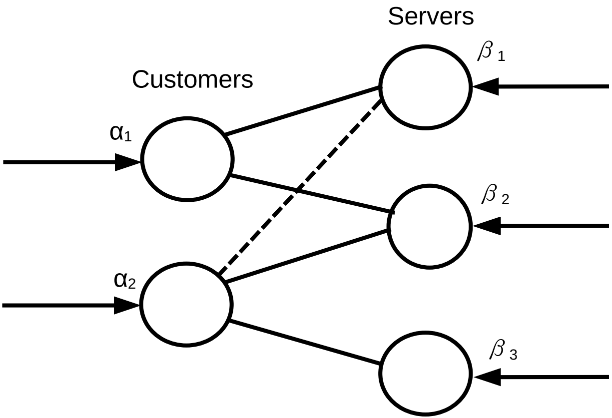

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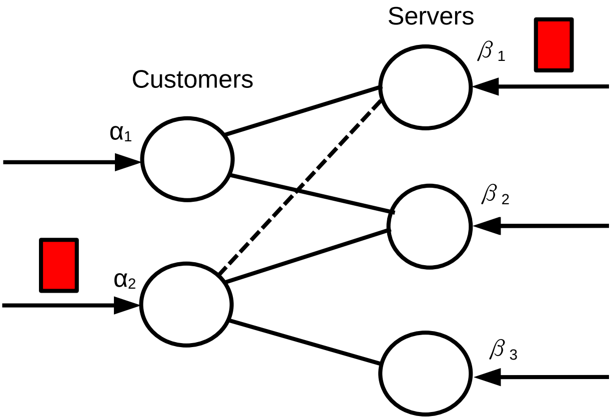

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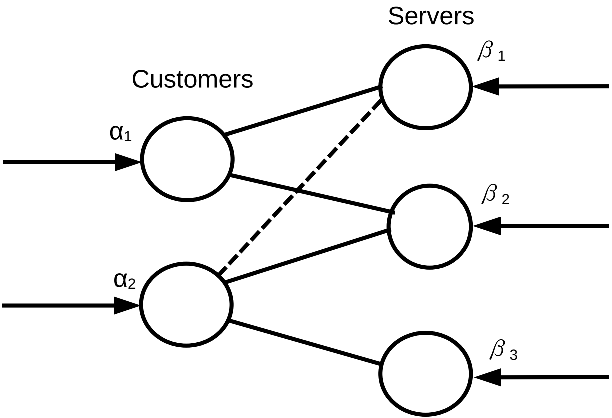

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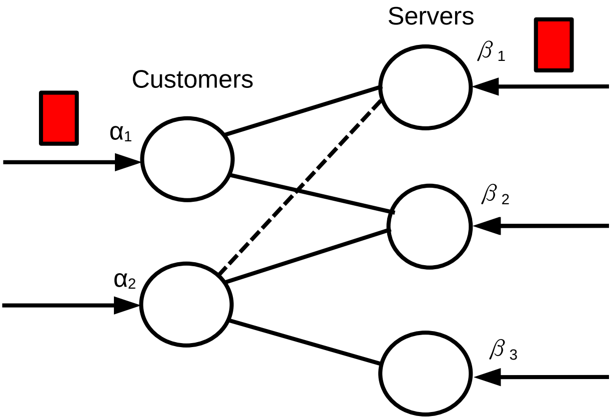

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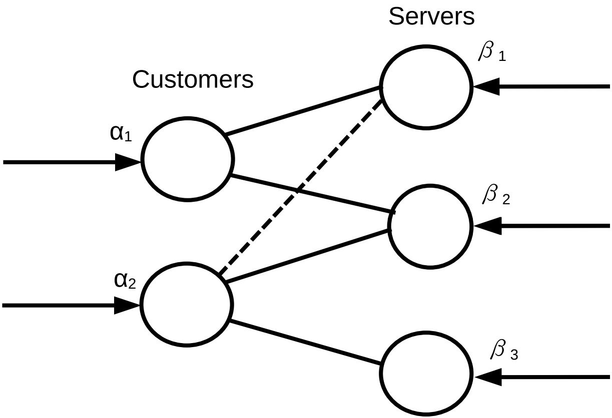

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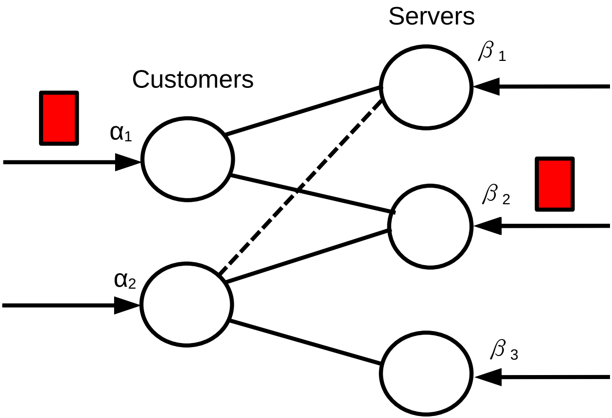

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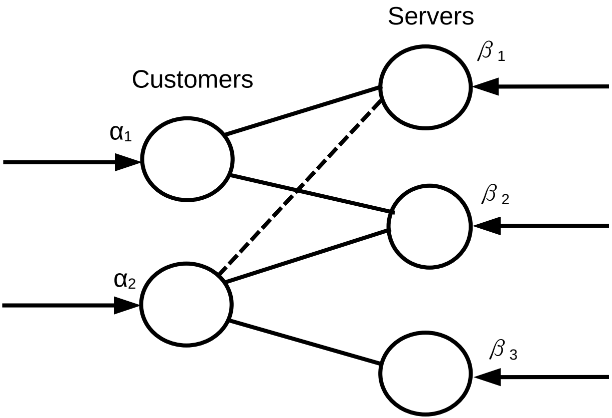

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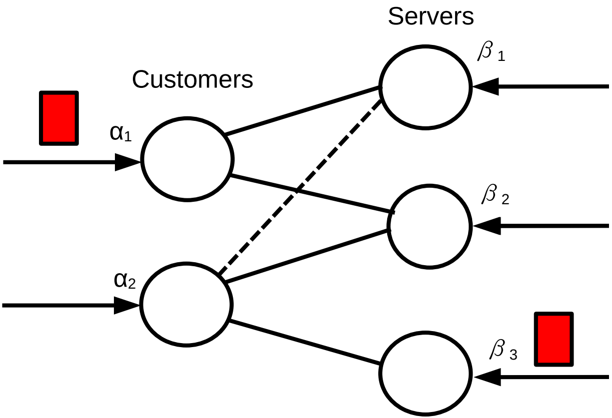

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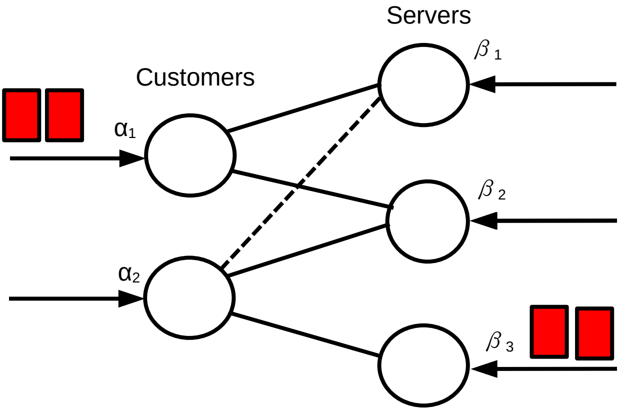

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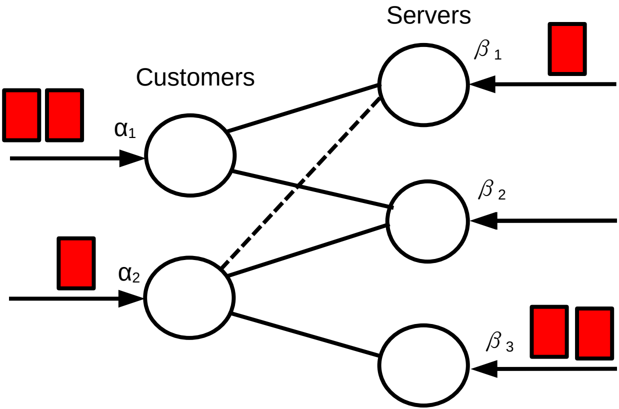

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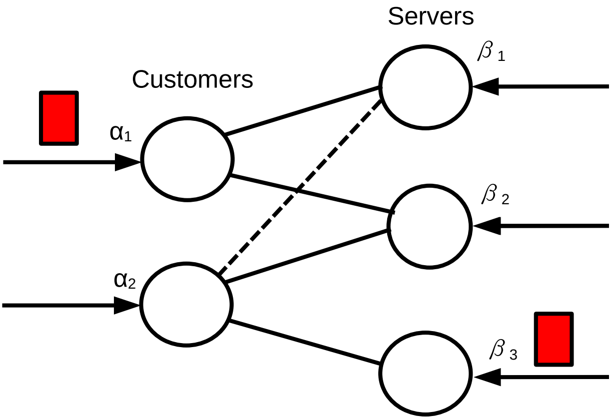

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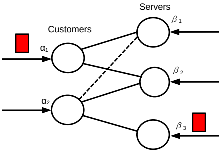

The couple (c_1, s_3) is the only possible unmatched pair of customer and server.

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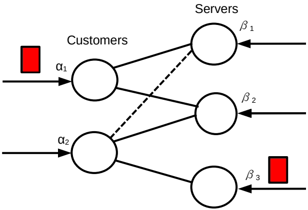

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The Markov chain is a birth-death process:

Birth probability: $\alpha_1 \beta_3$
Death probability: $\alpha_2(1 - \beta_3)$

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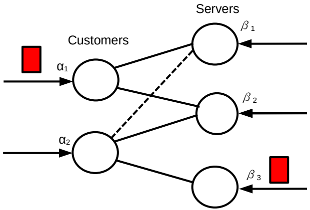

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The Markov chain is a birth-death process:

Birth probability: $\alpha_1\beta_3$
Death probability: $\alpha_2(1-\beta_3)$

Let $\rho = \frac{\alpha_1\beta_3}{\alpha_2(1-\beta_3)}$. If $\rho < 1$, the mean number of unmatched elements is $2\frac{\rho}{1-\rho}$

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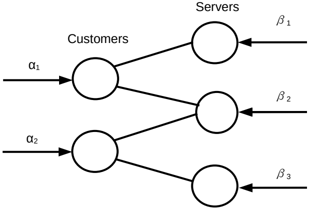

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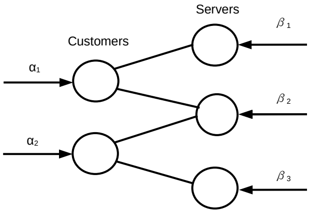

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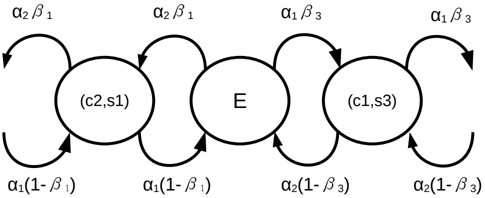
The Markov chain of unmatched couples is formed by two birth-death process connected in the empty state.



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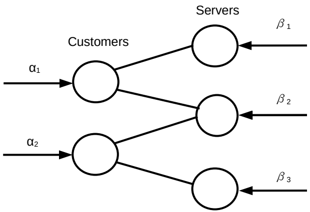

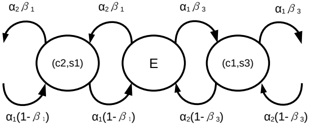

Let $\rho_1 = \frac{\alpha_1 \beta_3}{\alpha_2(1-\beta_3)}$ and $\rho_2 = \frac{\alpha_2 \beta_1}{\alpha_1(1-\beta_1)}$. If $\rho_1 < 1$ and $\rho_2 < 1$, the mean number of unmatched elements is

$$\frac{2(1-\rho_1)(1-\rho_2)}{1-\rho_1\rho_2}\left(\frac{\rho_1^2}{(1-\rho_1)^2} + \frac{\rho_2^2}{(1-\rho_2)^2}\right)$$

Performance Paradox Analysis

Objective: study the existence of the performance paradox

$$\frac{2(1-\rho_1)(1-\rho_2)}{1-\rho_1\rho_2}\left(\frac{\rho_1^2}{(1-\rho_1)^2}+\frac{\rho_2^2}{(1-\rho_2)^2}\right) \overset{?}{>} \frac{2\rho_1}{1-\rho_1}.$$

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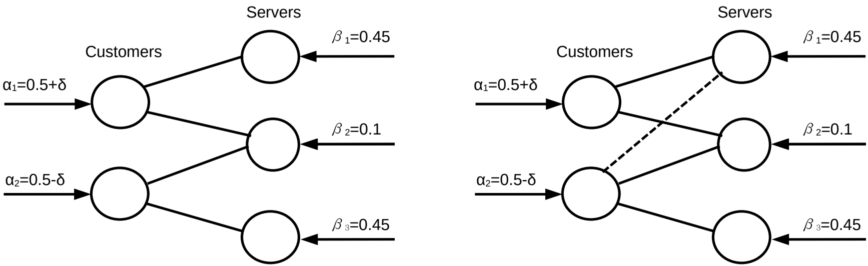

where $\delta \in (0, 0{,}05)$.

Performance Paradox Analysis

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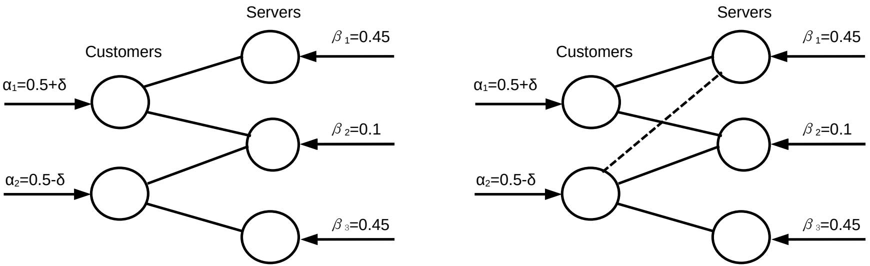

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Stability

We check that $\beta_1 < \alpha_1$ and $\beta_3 < \alpha_2$ for all $\delta \in (0, 0{,}05)$

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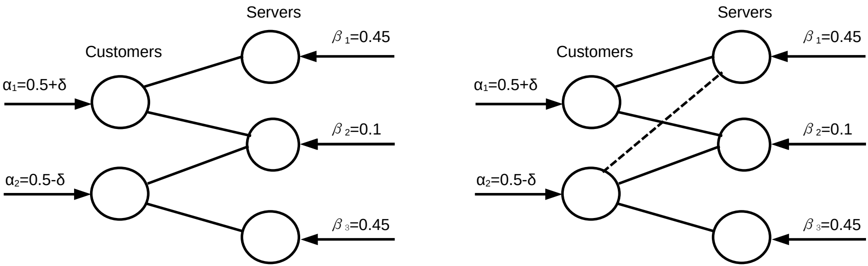

where $\delta \in (0, 0{,}05)$.

Stability

We check that $\beta_1 < \alpha_1$ and $\beta_3 < \alpha_2$ for all $\delta \in (0, 0{,}05)$ and also that $\rho < 1, \rho_1 < 1$ and $\rho_2 < 1$.

Performance Paradox Analysis

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$$\frac{99}{10}\frac{1+400\delta^2}{1-400\delta^2} \stackrel{?}{>} \frac{9(1+2\delta)}{1-20\delta}.$$

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$\Rightarrow$ When $\delta \to 0{,}05$, the mean number of unmatched elements tends to infinity

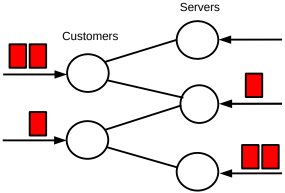
Proposition

When $\delta \to 0{,}05$, the difference due to the performance paradox tends to infinity.

Extensions

Matching policy: FCFM

Other matching policies lead to the same Markov chains: MaxWeight



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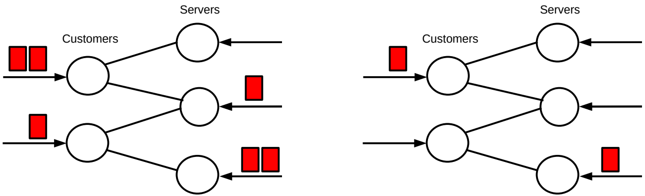

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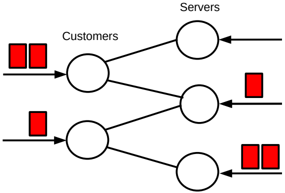

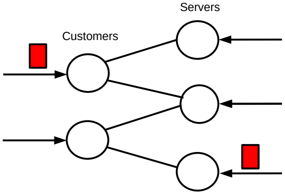

The W-shaped compatibility graph

The same results are obtained for any compatibility graph which is complete minus two edges

Outline

Conclusions and Future Work

Adding an edge in bipartite matching models might hurt the performance of the system

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Adding an edge in bipartite matching models might hurt the performance of the system

Questions for future work

- Is the existence of a performance paradox related to this particular compatibility graph?
- Can we provide sufficient conditions on the existence of a performance paradox in arbitrary compatibility graphs (and FCFM)?
- Does the performance paradox exist in multigraphs? And when we consider self-loops?

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- R. Caldentey, E. H. Kaplan, and G. Weiss. FCFS infinite bipartite matching of servers and customers. Advances in Applied Probability, 41(3):695–730, 2009.
- A. Busic, V. Gupta, and J. Mairesse. Stability of the bipartite matching model. Advances in Applied Probability, 45(2):351–378, 2013.

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- A. Busic, V. Gupta, and J. Mairesse. Stability of the bipartite matching model. Advances in Applied Probability, 45(2):351–378, 2013.

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Thanks and questions?

THANKS FOR YOUR ATTENTION

QUESTIONS?