A Mean Field Game with Interactions for Epidemic Models

Optimal Stochastic Control Approach

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Outline

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Introduction

Definition (Mean-Field Games)

Mathematical models for the study of the behaviour of a very large number of rational agents in interaction.

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Theorem (Lasry and Lions, 2006)

All Nash equilibrium converges as $N \to \infty$ to a Mean Field equilibrium. The equilibrium is unique under monotonicity conditions.

Assumptions:

- A1 Homogeneous players
- A2 Individual object action do not affect in the dynamics of the mass

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N players $\Rightarrow$ continuous players
Simplification of games and equilibria in the continuous limit

Introduction (cont.)

EDP approach to mean-field games: HJB and FP equations

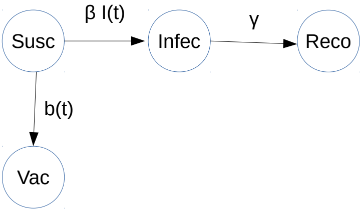
$$\begin{cases} -\nu\Delta u + H(x,\nabla u) + \lambda = V(x,m) \\ -\nu\Delta m - \operatorname{div}\left(\dfrac{\partial H}{\partial p}(x,\nabla u)m\right) = 0 \\ m > 0, \displaystyle\int m\,dx = 1 \end{cases}$$

$\Rightarrow$ Optimal stochastic control approach

SIRV Model

SIRV dynamics:

- Susceptible $\Rightarrow$ Infected: if it meets an infected with rate β
- Infected $\Rightarrow$ Recovered: with rate γ
- Susceptible $\Rightarrow$ Vaccinated: with rate $b(t)$



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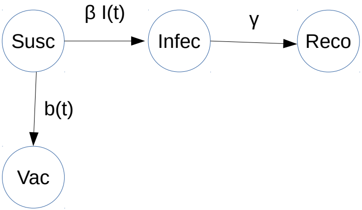


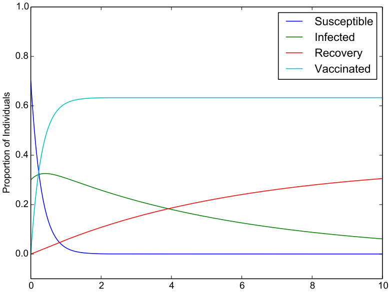
Some applications:

- Medicine
- Biology
- Computer networks: virus and adverts

SIRV Model

When $N \to \infty$:

$$\begin{cases} \dot{S}(t) = -\beta \cdot S(t) \cdot I(t) - b(t) \cdot S(t) \\ \dot{I}(t) = \beta \cdot S(t) \cdot I(t) - \gamma \cdot I(t) \\ \dot{R}(t) = \gamma \cdot I(t) \\ \dot{V}(t) = b(t) \cdot S(t) \end{cases}$$



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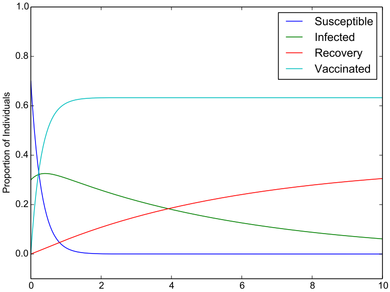

Classic MFG: interactions given only by the control

Mass dynamics depend on the control, Brownian motion...

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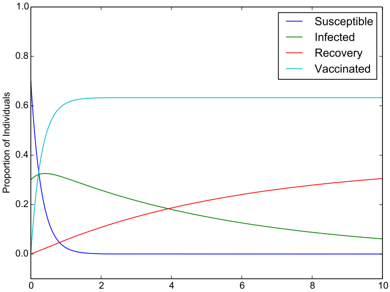

Classic MFG: interactions given only by the control
Mass dynamics depend on the control, Brownian motion...

Our model: mass dynamics depend also on $S(t) \cdot I(t)$
$\Rightarrow$ Mean-Field Game with Interactions

SIRV Model (cont.)

Vaccination policy: $b(t) \in [0, b_{max}]$
Vaccination cost: c_V
Infection cost: c_I

Obj: choose $b(t)$ to minimize cost

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Example: Hospital

- Decentralized $\Rightarrow$ each individual chooses how to vaccinate
- Centralized $\Rightarrow$ central agent decides when people take medicine

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Objective:

Compare cost of centralized and decentralized vaccination policies

Related work

No much literature of MFG with interactions

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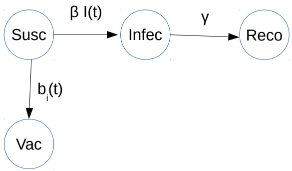
- (Laguzet and Turinici, 2015)
  Approximation:
  $P(X(t) = \text{infec}) = P(X(t) = \text{infec} \mid \text{no vac})$ and
  $P(X(t) = \text{vac}) = P(X(t) = \text{vac} \mid \text{no infec})$
- Our solution: No approximation

Outline

Decentralized Control

$X(t) \in \{S, I, R, V\}$ state of object $i \Rightarrow b_i(t)$



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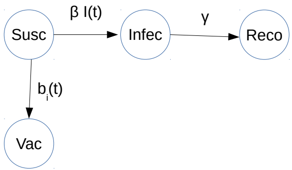

Generic player i: given $b(t)$, choose vaccination policy $b_i(t)$ to minimize his expected cost

$$\mathbb{E}\left(\int_0^T (c_V\ b_i(t)\ \mathbb{P}(X(t)=S) + c_I\ \mathbb{P}(X(t)=I))\ dt\right) \qquad (1)$$

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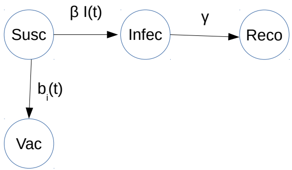

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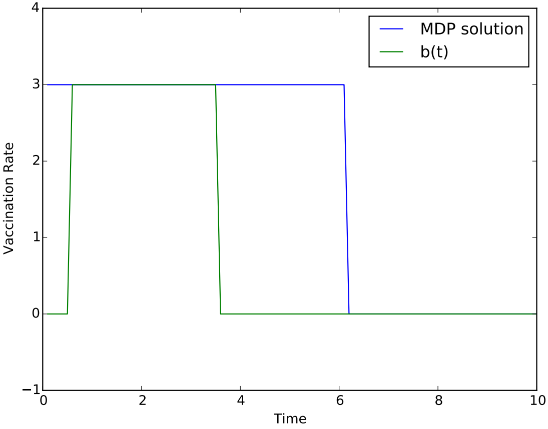
$$\mathbb{E}\left(\int_0^T \left(c_V\, b_i(t)\, \mathbb{P}(X(t)=S) + c_I\, \mathbb{P}(X(t)=I)\right)\, dt\right) \tag{1}$$

Continuous Time Markov Decision Process

Decentralized Control (cont.)

Proposition

For any $b(t)$, the solution of (1) is of threshold type



Mean-Field Equilibrium

Assumption: Homogeneous individuals $\Rightarrow$ solve (1) equally
Symmetric MFE

Definition (Mean-Field Equilibrium):

A vaccination policy is a symmetric MFE if and only if it minimizes (1) and it coincides with $b(t)$

Fixed point problem

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Solution of (1) of threshold type $\Rightarrow$ MFE requirements:

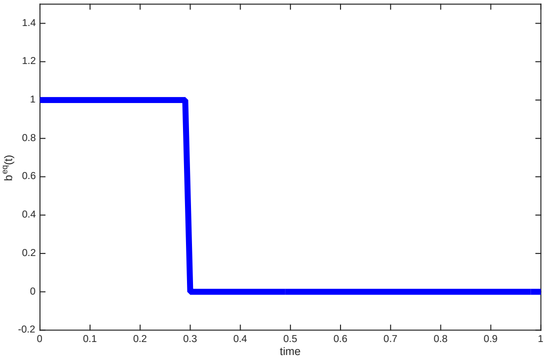
- $b(t)$ of threshold type
- Thresholds of $b(t)$ and of solution of (1) coincide

Mean-Field Equilibrium

Theorem

There exists a unique MFE and it is of threshold type.

Sketch of the proof: Monotonicity of MDP equations



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Centralized control

Definition: Global Cost

$$C_{Glo}(b(t)) = \int_0^T (c_V \; b(t) \; S(t) + c_I \; I(t)) \, dt$$

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$$b^{opt}(t) = \text{argmin}_{b(t)} C_{Glo}(b(t))$$

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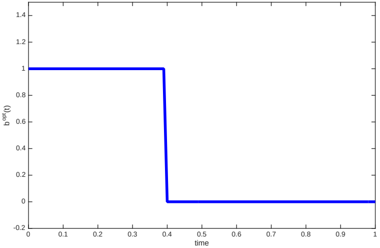
$b^{opt}(t)$ is of threshold type?



Centralized control (cont.)

Theorem

Global optimum is of threshold type

Centralized control (cont.)

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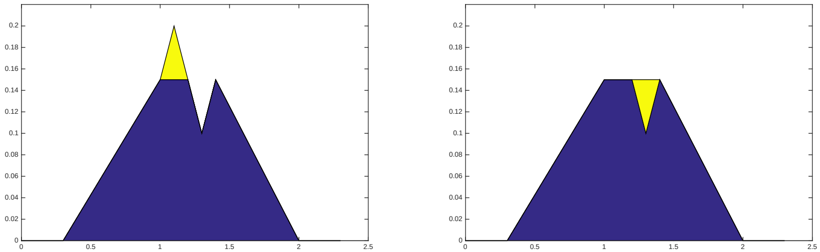
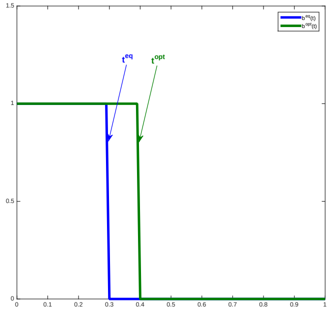
Sketch of the proof: Policy improvement



Left policy ⇒ less cost

Outline

Mechanism Design



Question:

Can we get that $C_{Glo}(b^{opt}(t)) = C_{Glo}(b^{eq}(t))$?

Mechanism Design

Observation:

For a fixed system parameters, except in the trivial cases, $t^{eq} < t^{opt}$. Therefore, $C_{Glo}(b^{opt}(t)) < C_{Glo}(b^{eq}(t))$

$\Rightarrow$ Change the model!

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Different cost of vaccination for decentralized and centralized problem

- Population vaccination cost: c_V
- Individual vaccination cost: $c'_V = c_V + p$

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p is positive, negative or zero?

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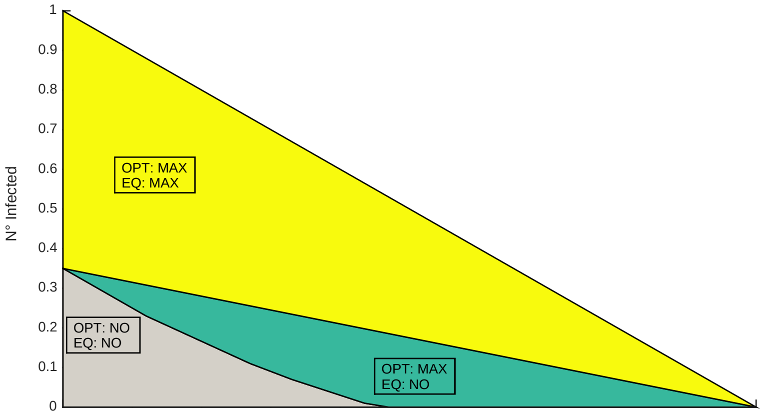
Dynamics and Thresholds



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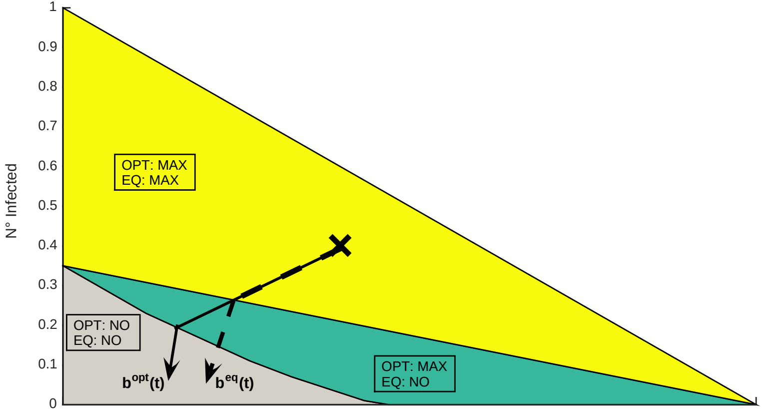

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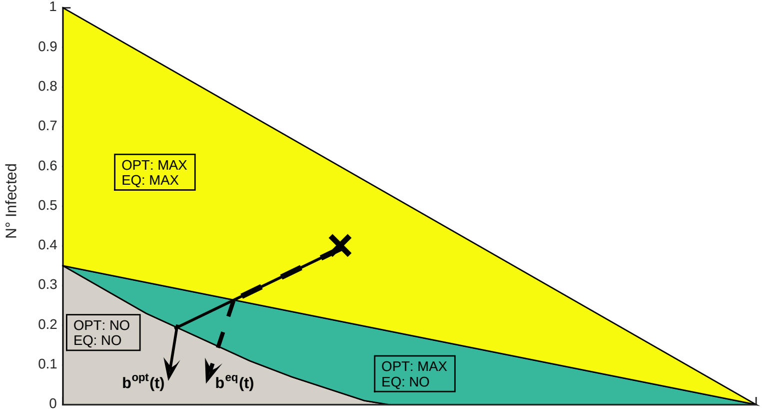
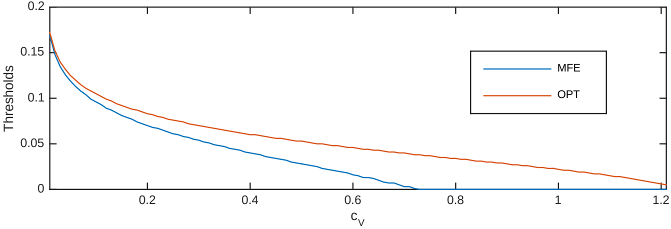
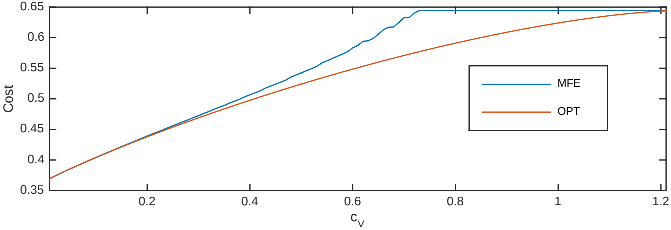


Conclusion

Except in the trivial cases, $t^{eq} < t^{opt}$

Varying c_V and Pricing Mechanism





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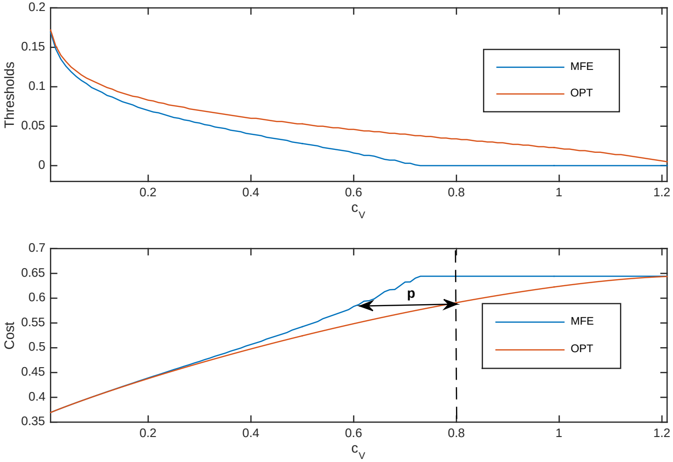

$c_V = 0.8 \Rightarrow p = 0.16$ (20% of c_V)

p: between 0% and 40% of c_V

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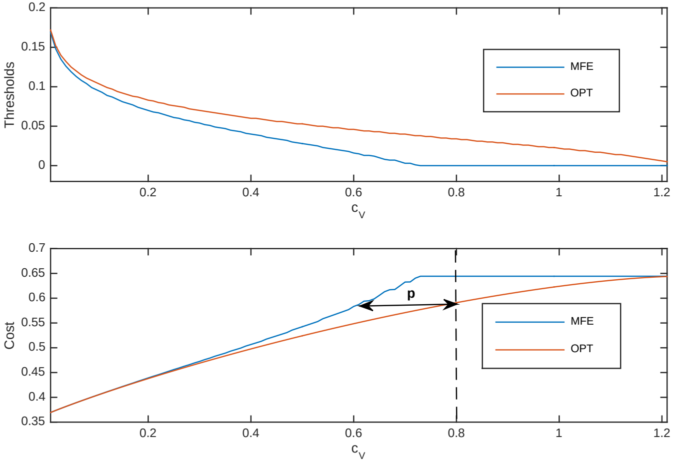

$c_V = 0.8 \;\Rightarrow\; p = 0.16$ (20% of c_V)
p: between 0% and 40% of c_V

Conclusion: $p < 0$

Vaccination to individuals must be cheaper!

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Conclusions

MFG ⇒ Optimal stochastic control

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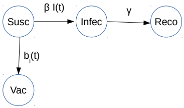
MFG $\Rightarrow$ Optimal stochastic control

Simple model: interactions and control



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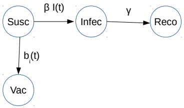

MFE is unique and of threshold type, as well as the global optimum

Pricing mechanism

Thank you

Questions?