On the Convergence to a Mean-Field Equilibrium

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The set of decisions such that no player has benefit of unilateral deviation

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There exists a solution for every non-cooperative game.

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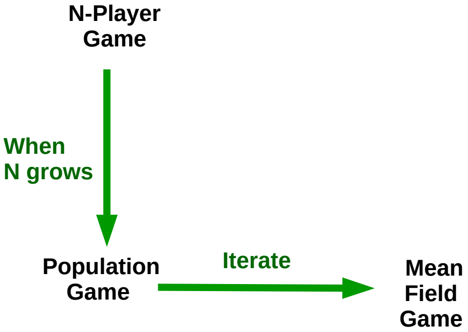
Mean-Field Games (Lions and Lasry)

Infinite number of rational objects in interaction.

Our question



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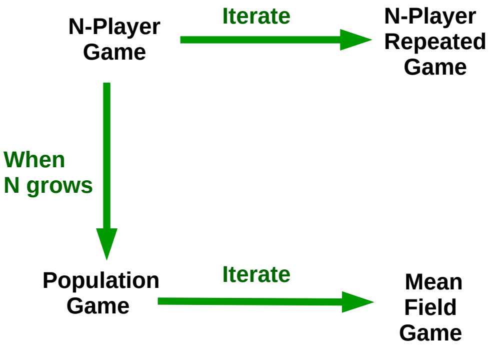

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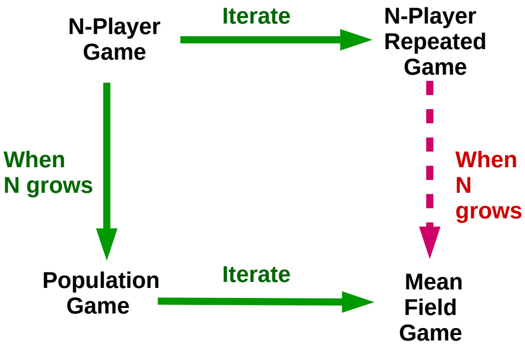

Outline

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Discrete Time

Infinite Homogeneous Players

- State: $\mathcal{S} = \{1, \ldots, S\}$
- Actions: $\mathcal{A} = \{1, \ldots, A\}$

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Mean-Field Equilibrium

Best Response to a

Set of strategies

$$BR(a) = \arg\min_{a^0} V(a^0, a)$$

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$$a^{MFE} \in BR(a^{MFE})$$

$\Rightarrow$ Fixed-point problem

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N-player game equilibrium

For all a'

$$V^N(a, a) \leq V^N(a', a)$$

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MFE existence

Continuity assumptions on $\mathbf{m}$

- $P_{ija}(\mathbf{m})$
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Theorem

Any discrete time mean-field game with discounted cost that satisfies the continuity assumptions has a mean-field equilibrium.

Best-response has a fixed-point: Kakutani

Convergence

Every repeated game Nash equilibria converge to a MFE?

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Player Strategies of N-player repeated game

- Local: depend on state and time $\Rightarrow a(i, t)$
- Markov: depend on $\mathbf{m}$, state and time $\Rightarrow a(i, \mathbf{m}(t))$

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Theorem

Every local equilibrium converges to a mean-field equilibrium.

Proof: $V^N(\pi',\pi) \to V(\pi',\pi)$ when $N \to \infty$

(H. Tembine, J.-Y. L. Boudec, R. El-Azouzi, and E. Altman. Mean-field asymptotics of markov decision evolutionary games and teams. GameNets' 09.)

Prisoner's dilemma

$\mathcal{S} = \mathcal{A} = \{C, D\}$

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$\Rightarrow m_C + m_D = 1$

$$c(i, m) = \begin{cases} m_C + 3m_D & \text{if } i = C \\ 2m_D & \text{if } i = D \end{cases}$$

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Static game equilibrium $\Rightarrow$ Always D

- MFE?
- Repeated Game NE?

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$a^D = BR(a)$

The best response to any action a is always D.

$$V(i,m) = (1-\delta)\sum_{t=0}^{\infty} \delta^t \left(x_C(t)\cdot(m_C(t)+3m_D(t)) + x_D(t)\cdot 2m_D(t)\right)$$

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Action a^*

- First k rounds, play D
- From k, play C if all play C
  Otherwise, play D for ever

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- First k rounds, cost increases
- From k, if play D, an immediate advantage $\Rightarrow$ punished until the end

What happened?

Theorem (Folk Theorem)

Let G be a symmetric matricial game, and let V^* be the cost under the strategy that repeats the Nash equilibrium of the static game G. Then any feasible cost V smaller than V^* is the cost of an equilibrium of the repeated game if the discount factor is large enough.

Reward and Punishment Principle

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The Folk Theorem do not scale

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Continuous Time and Discounted Case

$Q_{ija}(\mathbf{m})$: transition rate matrix
Assump: $Q_{ija}(\mathbf{m})$ continuous in $\mathbf{m}$

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Markov strategies

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Markov strategies do not converge
$\Rightarrow$ Prisoner's dilemma

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Static Nash Equilibria: D and P

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Static Nash Equilibria: D and P

Repeated Game Nash Equilibrium:

- if $t < 1$, play C if $m_C = 1$, play P otherwise
- if $t \geq 1$, play D if $m_P = 0$, play P otherwise.

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Conclusions

Convergence:

- Local strategy $\rightarrow$ MFE
- Markov strategies do not

Discrete and continuous time

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MFE existence:

- Discrete and continuous time
- Discounted and finite horizon

Future work

Convergence:

- Best class of actions $\Rightarrow$ convergence

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Full version available:
Josu Doncel, Nicolas Gast, Bruno Gaujal. Mean-Field Games with Explicit Interactions. `https://hal.inria.fr/hal-01277098`